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Constitutive Behavior Based on Crystal Plasticity

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1. INTRODUCTION

Constitutive equations are the vehicle by which our knowledge of material behavior enters into engineering design. At the very least, they should be sufficiently accurate. This could be—and frequently is—achieved by an empirical description based on data obtained under conditions that essentially duplicate those of the specific application. Of more general usefulness are relations that can be applied under a wide range of conditions and for many materials, containing a number of materials parameters (the fewer the better), which can be measured in simple tests. Such general relations can be expected to be found only if they fulfill two conditions: they must be phenomenologically sound; and they should be based on as much of the underlying physics as can be ascertained with some confidence. The closer the phenomenological description reflects the actual physical processes involved, the further it can be extrapolated beyond the range of variables for which it was measured.

Phenomenological soundness reflects, *inter alia*, an appropriate choice of variables, and a formulation that exhibits the proper invariance against arbitrary frames of reference. For example, it recognizes that the material responds to stresses, not forces (thus separating parameters of the geometry from those of the material), and that the stress is a second-rank tensor; it takes proper account of the changes in geometry with finite deformations, usually by the use of matrix descriptions. These problems are by no means trivial, but they

AUTHOR'S NOTE:

Much of this material was later treated as Chap. 8 of "Texture and Anisotropy," Kocks, Tome, Wenk, eds. (1997, Cambridge University Press). Section 2 of the present article is deemed still essential.

are solvable. More subtle are judgments as to whether one should use history or state variables, integral or differential descriptions, and the like.

We will address some of these questions briefly but will, in the main, oversimplify the phenomenological aspects in order to concentrate on the material properties. Thus, we will inquire into the behavior of a *material element* (a convected volume element) under a macroscopically uniform stress during a macroscopically uniform, *infinitesimal* increment of strain. These (local, average) stresses and strain-increments are supposed to be related to surface tractions and surface displacements by the standard methods of solid mechanics. For this purpose, the medium in which the element is embedded is considered non-dissipative—though the element itself is essentially dissipative. By this convention, the local stresses and strain-increments become, in effect, the ‘applied’ variables (regardless of which is viewed as the independent and which as the dependent one). Their product is the work done by the environment on the material element under consideration, representative, in the end, of the lowering of weights at the surfaces.¹

We will regard the material element as being at *constant temperature*, on the time scale for which the behavior is described. The *material response* is then principally the relation between the stress and the strain-increment. In addition, the stress increment, time-rates of change, and other variables may enter under certain circumstances.

The prime lesson to be learned from materials science is that there is not one material response, one ‘mechanical equation of state’, or even one set of differential constitutive equations. If such a completely general formulation were attempted, it would be too complicated to be of any use. A more effective approach is to look for *classes of materials*, *regimes of variables*, and *aspects of behavior*, for which a ‘universal’ constitutive description can be found. For example, in the present treatise, we will concentrate on polycrystalline, single-phase metals of cubic lattice structure that have been plastically deformed by, say, 1–100%, at temperatures between about 20 K and one-half to two-thirds of the melting temperature, at strain rates between about 10^{-7} and 10^3 s^{-1} . With this restricted (though very broad) ‘interest space’, deformation is governed by *crystallographic slip* in the grains of the material element or, on a finer scale, by *dislocation glide* and *dislocation storage*. This assessment of the physical mechanisms allows

one to formulate a meaningful set of stress/strain relations, with respect to both the kinetics and the multi-axial behavior.

An important input from an understanding of the physical mechanisms is the provision of *diagnostic tools* to assess whether a specific material under specific conditions in fact falls within the assumed 'regime'.^{1,2} In Section 7 we will give some examples; in particular, it will be outlined how one can assess whether a material behaves, in its macroscopic properties, like a 'single-phase' or a 'multi-phase' material, which has important implications for the hardening rule to be expected.

A final decision one has to make is which *aspects of behavior* to include. Again, if one attempted to condense *all* aspects of the mechanical behavior into one general set of constitutive relations, these would quickly become unmanageable. This is the point in any complex problem where judgment becomes of paramount importance. For the purposes of the present treatise, considering the 'interest space' circumscribed above, it is our judgment that a sufficiently self-contained description of *plasticity* can be obtained by ignoring elasticity and anelasticity, unloading and reverse-loading effects, inertial effects and body-forces, and energy storage. We will give some justifications for this judgment in Section 2. The judgment concerns material properties only; for *calculational* purposes, an inclusion of elasticity is sometimes necessary (when pure plasticity relations cannot be inverted), and body forces are sometimes used explicitly as an algorithmic tool, etc.

Our primary concern will be with three aspects of material behavior:

- the *kinetics* of flow under the influence of thermal activation, which is well described elsewhere^{1,3} and will only briefly be summarized in Section 3;
- the influence of polycrystal plasticity on the *multi-axial* stress-strain relations for *anisotropic* materials, for which we present new results in Section 4;
- the description of the *evolution* of the state parameters, which is given for both texture and substructure evolution in Section 5, including some new proposals for treating a specific second state parameter.

To round out these primary concerns, we will discuss various meanings of the term 'internal stress' in Section 6, and assess them with respect

to the necessity or usefulness of introducing such an extra parameter. Finally, in Section 7, we summarize diagnostic procedures to establish the type of behavior that controls a given material in a given regime of the variables, and summarize the constitutive relations for the interest sphere emphasized in this article. This leads to some general recommendations in Section 8.

A recurring theme in constitutive relations is scaling laws. We will give some general guidance to various stress and temperature scaling parameters. In addition, we discuss briefly the fundamental question of *scale*: the size of a meaningful material element (Section 3.1.1).

A major theme of this book is 'unified' constitutive equations for 'plasticity' and 'creep'. In the physical theories of plastic deformation, this unification exists *ab initio*: whether the strain (rate) is prescribed and the stress is measured, or the stress is prescribed and the strain (rate) measured—the material response is the same, it must be independent of the boundary conditions.

The material response is also independent of the history; it is entirely determined by the current (micro-)structure, regardless of which path was taken to get there. This may be called the 'article of faith' of material scientists, and it will be assumed throughout this work: *current behavior depends only on the current state*. The current rate of evolution of the state is one aspect of current behavior.

2. SOME IMPORTANT REALITIES

For problems as complicated as plasticity, there is no hope of ever finding a 'correct' solution, from first principles, even for a restricted interest space. The most important decisions are made before one writes down the first equation: namely, what to consider important and what to neglect if necessary. It is not only approximations at the solution stage that are made (as anywhere in physics), but also judgments at the problem-setting stage. To be as wise as possible in making these judgments, it is imperative to have most of the basic realities in mind.

2.1. Uniaxial Monotonic Deformation

2.1.1. Yield

Figure 1(a) shows the beginning of a typical stress/strain curve. It is drawn on a scale that emphasizes the transition from elastic to plastic

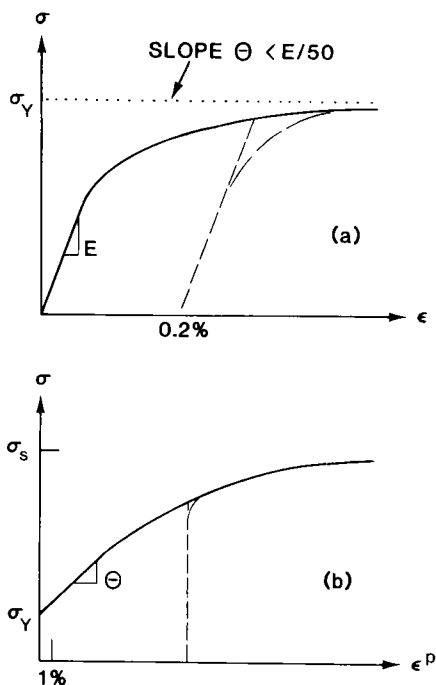


Fig. 1. The same stress/strain curve drawn schematically on two scales: (a) to emphasize the elastic-plastic transition, which is typically completed by about 5 times the elastic strain; (b) to emphasize the fully plastic state and strain hardening. The dashed lines show unloading/reloading behavior.

behavior. For about half of the stress range shown, the behavior is linear-elastic (modulus E). The beginning of deviations from linearity coincides, for most materials,[†] with the beginning of plasticity: upon unloading, there would be a permanent 'offset'. This first deviation from nonlinearity is sometimes used as the definition of 'yield'; more succinctly, it is called the 'proportional limit'. Since the first deviation from linearity is a matter of the resolution of the experiment, a definition in terms of the magnitude of the offset is often preferred. A typical choice for 'microyield' is an offset strain of 10^{-5} .

[†] Nonlinear elasticity of the deviatoric components is important only in materials of exceptional strength.

The slope of the stress/strain curve then drops precipitously: typically, it changes by one or two orders of magnitude as the stress increases by a factor 2. For this reason, it is easy to define an *asymptotic* behavior: on the scale of Fig. 1(a) it is a horizontal line; this stress is normally defined as the *yield stress* (or yield strength, or macroyield stress, or flow stress). Another common definition uses an offset of 0.2%; this is, in practice, for many materials, equivalent to the asymptotic definition, because yield is so sharp. (A more general definition would be an offset of 5 times the elastic strain: then the material is essentially fully plastic.)

The asymptote is not exactly horizontal. Fortunately, it is approximately *straight* for many materials (single-phase, at low temperatures), with a slope of about $E/50$. Then, *back-extrapolation* is a straightforward procedure (or rather, a straight-backward one). The loading slope measured in a typical experiment is less than E , because of a finite machine compliance—but usually larger than $E/5$. Thus, the ratio between the two recorded slopes is usually larger than 10. This permits an accurate determination of the point of intersection between the two lines: the ‘yield stress’.

In two-phase materials, or at high temperatures, the initial strain-hardening behavior may not be linear. In such cases, a plot of the *square* of the stress against strain may well give an approximately straight asymptote near yield, and may then be used to determine a back-extrapolated yield stress.^{2,4} Alternatively, one may here resort to a 0.2%-offset definition.

2.1.2. Strain Hardening: the Flow Stress

Figure 1(b) is a stress/strain curve drawn to a different scale; here the elastic slope appears infinite—but what was called the (‘horizontal’) asymptote above now appears with a definitely finite slope, displaying the effects of strain hardening. It tends to its own asymptote, at perhaps 2 to 20 times the yield stress: the ‘saturation stress’ or ‘steady-state stress’. (Actually, in many materials the asymptote is not truly horizontal, but its slope is typically another order of magnitude lower.⁵⁻⁸)

Figure 1(b) also shows the effect of unloading, after some significant plastic strain, and reloading. There is again an elastic-plastic transient, typically much sharper than at initial yield—but then the curve asymptotically approaches the continuous one. Thus, the back-extrapolated yield stress upon reloading is equal to the stress last

reached before unloading; this is usually called the *flow stress*. The *strain hardening curve* is then also the *locus of all flow stresses as a function of plastic prestrain*.⁹ This differs, in principle, from the 'stress/strain' curve (Fig. 1(a)) in that the elastic-plastic transition region is not treated as 'strain hardening'. In practice, on any scale on which strain hardening can be seen (Fig. 1(b)), the two diagrams are indistinguishable, and so are the total strain and the plastic strain; this is how we will use 'strain' throughout this paper.

2.1.3. Transients

There are principally two types of transients in addition to the elastic-plastic transition described above.¹⁰ Figure 2(a) illustrates a *yield drop*; it occurs in some materials, such as mild steel, at the beginning of testing (after an anneal), and in many other materials after unloading and reloading (with or without deliberate aging). Its length is typically much less than 1% strain, unless a Lüders band propagation is associated with it, in which case it may be of the order of 1% strain. We choose to ignore such transients here, and continue to consider 'yield' to occur at the stress obtained by back-extrapolating

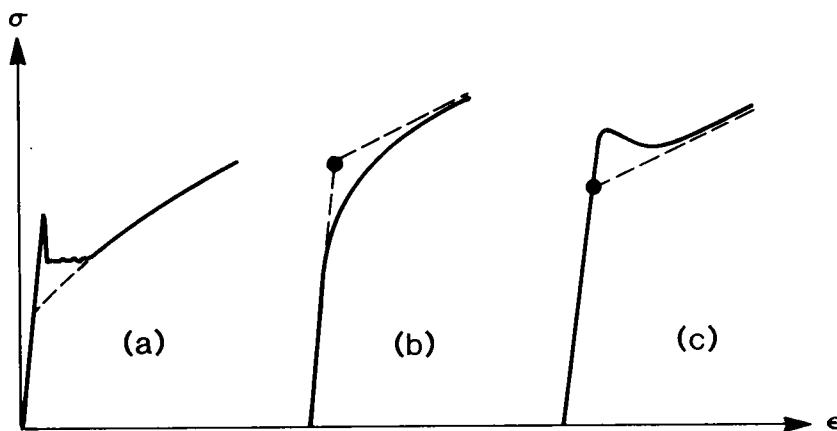


Fig. 2. Some commonly observed loading transients: (a) sharp yielding in aged materials; (b) gradual yielding after recovery or a large increase in strain-rate or a reversal of the straining direction; (c) 'work softening' after a large decrease in strain-rate or a general straining-path change. (Similar behavior is observed in essentially dislocation-free materials.) None of these transients lasts longer than a few percent strain.

the steady stress/strain curve—either to the ‘lower yield point’ or, if feasible, beyond that to the elastic line.³

The other type of transient is longer (typically about 3% strain) and often difficult to ignore or extrapolate through. It occurs after *path changes*, and primarily when the strain-hardening rate is low (due to dynamic recovery). For example, after an increase in strain rate or decrease in temperature, it typically looks like Fig. 2(b), after the reverse path change like Fig. 2(c) (this is called ‘work softening’). A change in ‘stress path’ (i.e. in the ratios of the components of the stress and/or strain-rate tensors; see Section 2.2) may cause either one of these behaviors.

A special case of a path change is *stress reversal*: after previous straining in the reverse direction, a new loading curve looks about like Fig. 2(b). In a *microyield* definition, this amounts to a substantial lowering of the ‘yield stress’ upon reversal, and this is often referred to as the ‘Bauschinger effect’. It is seen that, from a macroscopic plasticity point-of-view, such a description does not capture the essence of the effect: the Bauschinger effect is primarily a *transient in the strain-hardening behavior*. The asymptotic behavior after the transient may or may not coincide with that without reversal, depending both on the details of the chosen description and on the type of material. We shall make use of this material dependence as a diagnostic tool to identify two-phase materials in Section 7.

2.1.4. Rate Sensitivity

The rate sensitivity of the flow stress may be defined as

$$m = \left. \frac{\partial \ln \sigma_Y}{\partial \ln \dot{\epsilon}} \right|_T \quad (2.1)$$

where σ_Y may be the yield stress or the flow stress in a prestrained state of the material, as defined in Fig. 1(b). (However, it is *not* the flow stress attained after the same strain at different strain rates; see Section 5.2.4.) This ‘isostructural’ rate sensitivity m is usually negligible in magnitude, of order 0.01.^{11,12} This fact has led to the common idealization of a rate-independent material, which is often useful. One case in which the rate sensitivity becomes of paramount importance is when it is *negative*: this causes instabilities (unless some other stabilizing factor intervenes).³

There is another rate sensitivity: that of the flow stress in the limit of

steady state (i.e. zero strain-hardening, which is controlled by the rate sensitivity of strain hardening. It is commonly characterized by a 'stress exponent' n (the inverse of a rate sensitivity):

$$\frac{1}{n} = \left. \frac{\partial \ln \sigma_s}{\partial \ln \dot{\epsilon}} \right|_T \quad (2.2)$$

$1/n$ is always much greater than m ; it increases more-or-less linearly with temperature,¹³ from values around 0.03 near room temperature (strongly dependent on the specific material) to near 0.3 at high temperatures. This rate sensitivity is responsible for a significant increase in stability against necking.³

While all of these rate sensitivities may be quantitatively neglected in many applications, the *principle* of rate sensitivity is useful in two respects: (a) it eliminates phenomena such as bifurcations and vertices on yield surfaces that are really artifacts of the idealization of a *strictly* rate independent material;⁹ and (b) it serves as a potent diagnostic tool to identify physical mechanisms.¹ We shall use these two effects of rate dependence in Sections 4 and 7, respectively.

All the rate sensitivities in the regime of interest we have circumscribed are intimately linked to temperature sensitivities, because they are due to thermal activation. We will summarize these relations in Section 3.

2.1.5. Pressure Dependence and Volume Changes

Yield stresses and flow stresses are almost always proportional to elastic constants—and elastic constants are pressure dependent. Thus, yield is, in principle, pressure sensitive. In many cases, experimental observations of pressure dependence are in quantitative agreement with this explanation.¹⁴ It makes the yield strength go up with pressure.

Other pressure effects could be due to a change in dislocation core configuration with pressure, and a consequent change in the lattice resistance ('Peierls stress'). In this case, one would expect the yield stress to decrease as the pressure increases (since the cores get wider).

Neither of these pressure effects causes any plastic volume change. Volume changes are, however, possible through the accumulation of defects with deformation: dislocations themselves as well as vacancies (or voids) and self-interstitials cause some increase in the volume. An applied pressure could therefore decrease the rate of accumulation of these defects and thus influence *hardening* processes.

All of these effects are usually quite small and shall be neglected in the present treatise. They would be significant in materials of exceptional strength (such as high-strength steels¹⁴) or exceptional elastic nonlinearity (such as polymers¹⁵), or at very high pressures, or very large strains (when void generation becomes important).

Finally, we emphasize that *elastic* volume changes (as opposed to deviatoric elastic strains) can be quite large when the pressure is great: in strong shocks, the volume may decrease by a factor 2.

2.1.6. Energy Storage

The accumulation of dislocations during plastic deformation, and the possible changes in microstructural details such as precipitate size, shape, degree of order, etc., require some of the plastic work to go into stored energy. A simple estimate shows that the rate of energy storage divided by the rate of work done should be of order θ/μ :¹⁶ the shear hardening rate (in crystallographic slip) divided by the shear modulus. This quantity is always less than 0.01 in single-phase materials of cubic lattice structure, although it may be quite high (especially at low strains) in two-phase materials and polycrystals of hexagonal or lower symmetry materials. Experimental determinations of the energy stored after deformation typically give about 3% of the work done.

The rate of energy storage provides a 'thermodynamic threshold'⁹ for flow: a minimum stress that must be applied to furnish the energy to be stored. Due to the above estimates, this stress can be safely neglected for all but the very lowest stresses, at high temperatures (for the materials of interest here). We shall therefore consider plastic processes to be essentially dissipative.

2.1.7. Conclusion

A good approximation for a description of the plastic response of many materials under many conditions is to treat them as *essentially* rigid-plastic, purely dissipative, rate independent, pressure independent, and isochoric: this we shall do in the present text. However, none of these idealizations is *strictly* true; thus, they should not be taken seriously when they lead to *qualitatively* different behavior. With these restrictions, we will emphasize the behavior under *multiaxial* deformations. We will also treat in some detail the description of *evolution* (including its rate dependence).

2.2. Multiaxial Deformation

2.2.1. The Yield Surface

The fact that yield is sharp and rate-insensitive makes the concept of a *yield* condition useful (as opposed to a treatment in terms of a plastic *modulus*, decreasing with strain, or a plastic *viscosity*, increasing with strain rate). This is true particularly when 'yield' is defined as the *asymptotic* behavior discussed in Fig. 1(a); for *small*-offset ('micro-yield') definitions, the concept of a yield surface is questionable—we shall return to this below (Section 2.2.3).

The yield condition is graphically described by a surface in stress space. Since the stress is a symmetric second-rank tensor, 'stress space' is six-dimensional; since the hydrostatic component of stress is considered irrelevant (or essentially irrelevant) for plastic flow, one usually uses the *five-dimensional* deviatoric stress space. In such a space, it is convenient to describe the stress as a *vector* (in the sense of a 5×1 matrix, not in the sense of a first-rank tensor). The yield surface is, then, the locus of yield stresses for different directions of the stress vector (i.e. different ratios of the stress tensor components). Note that the 'yield stresses' do not transform as a tensor; their directional dependence is described by the yield surface. It is for this reason that the term '*yield strength*' is sometimes preferred over '*yield stress*'.

Symmetry considerations can reduce both the dimensionality and the extent of the stress space in which the yield surface must (at least) be given for a complete description. For isotropic materials, as is well known, a 60° sector of the plane of principal deviatoric stresses suffices; if there is no sign-dependence of yield, a 30° sector suffices. In this sector, the demands of convexity (see further below) limit the possibilities severely. Two commonly used assumptions are the 'Tresca hexagon' (which is sometimes realistic, e.g. for the lower yield point in mild steel) and the von Mises circle. It must be emphasized that the *circle* in *stress space* is not a consequence of isotropy; it is an assumed shape of the yield surface within the bounds allowed by isotropic symmetry.

A point often made, and true, is that, were it only to discover the correct shape of the yield surface for isotropic materials, a great effort would not be worthwhile, since the differences are so small. The fact, however, is that experimental yield surfaces often depart very substantially from those allowed for isotropic materials—because many

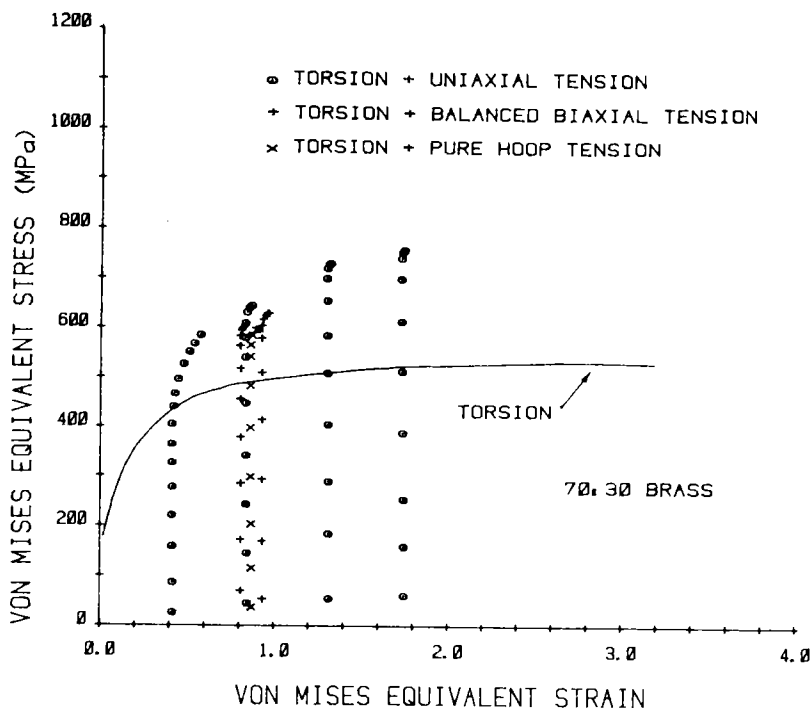


Fig. 3. Probes in different straining directions after various prestrains in torsion; long tubes of brass.¹⁷ The ratio of yield stresses in any one state is considerably greater than allowed by an assumption of isotropy.

materials (e.g. all that have been heavily deformed) are *not* isotropic. An example is shown in Fig. 3,¹⁷ which represents probes in various directions of stress space after various (large) amounts of prestrain in torsion. It is plotted so that, were the von Mises criterion valid, the reloading yield points would have to fall on the continuous curve. The departures are quite substantial.

2.2.2. Equivalent Stress and Equivalent Strain

The size of the yield surface defines a scalar parameter, the 'equivalent stress' (or 'effective stress'), σ_e . Its change with strain may be used to describe a form of *hardening* (called 'isotropic hardening'; see below). An 'equivalent strain' is usually defined as its work conjugate (so that $\sigma_{ij} d\epsilon_{ij} = \sigma_e d\epsilon_e$). An unfortunate experimental fact, however, is that a

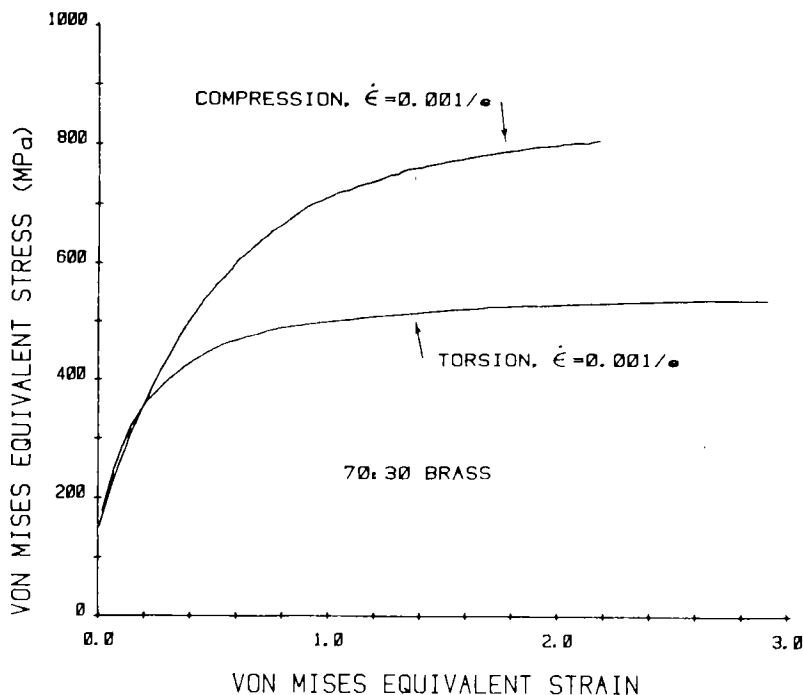


Fig. 4. Continuous straining in torsion and compression; brass.¹⁷ The comparison is made on the basis of the von Mises equivalent stress and strain; it does *not* serve to unify the plots.

plot of equivalent stress versus equivalent strain does *not* usually give a unique hardening law for all deformation modes. Figure 4 shows this in an example for torsion and compression in brass, using the von Mises definitions of equivalent stress and equivalent strain.^{17†} There are two plausible causes for this discrepancy: one that, as stated above, few metals are nearly isotropic; but more fundamentally that there is no reason to expect the state of the material to be the same when the 'equivalent strain' achieved along two different strain paths is the same (see Section 5.1.3).

† Sometimes the terms 'equivalent stress' and 'equivalent strain' are used as *identical* to those according to the von Mises postulate; we will discuss this in detail in Section 4.2.1.

2.2.3. Hardening Rules

This term is commonly used to describe changes in the *yield surface* with strain. (The *rate of change* with strain is 'strain hardening' or 'work hardening', discussed above). The two most common hardening rules are: 'isotropic hardening', a change in the *size* of the yield surface only (all dimensions being scaled by the same constant—whether the material is plastically isotropic or not); and 'kinematic hardening', a change in *location* of the yield surface only (which requires a *tensor* to be specified).

Figure 5 shows the results of an experimental probe of the yield surface after a significant prestrain in torsion.¹⁸ 'Yield' was evaluated both in the small-offset definition ($\epsilon = 5 \times 10^{-6}$) and as the back-extrapolated value. It is seen that the latter deviates little from 'isotropic hardening'. Microyield, on the other hand, involves a change

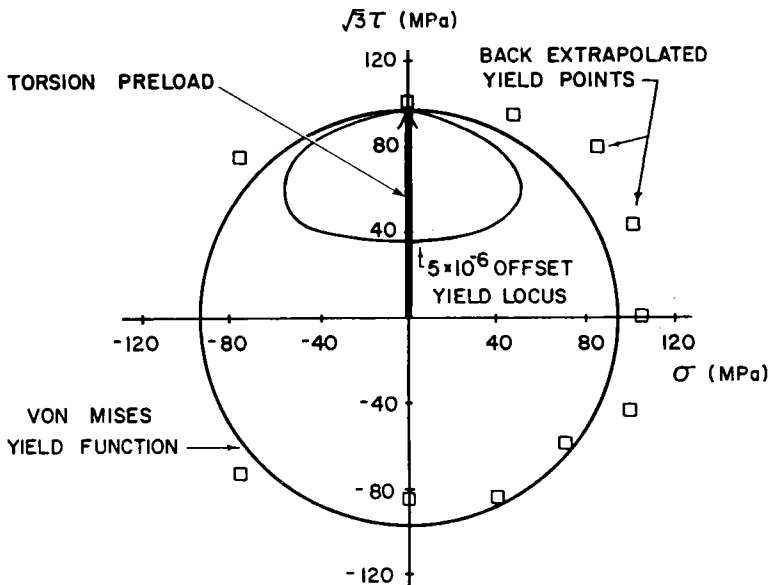


Fig. 5. Experimentally determined yield loci, after a pre-shear-strain of 0.5 in torsion of a long tube of 1100 aluminum.¹⁸ The last stress reached during prestraining is shown as a heavy arrow. The small oval 'yield locus' was taken at a very small offset after unloading only into its interior. The von Mises circle is shown for reference. The squares show back-extrapolated yield points. (Yield stresses at 0.2% offset were the same.)

in size, location and, in fact, *shape* of the 'yield surface'. Another way of describing the small-offset yield surface is that it is almost like the macroyield surface except for a drastic reduction in 'yield' in the reverse direction (to the point of occurring already during unloading, in this case). This is in congruence with the description of the Bauschinger effect we gave in Section 2.1.3: it is primarily a work-hardening transient.

Another hardening rule has been derived from 'slip theory':¹⁹ here, only the direction of prestraining hardens, developing a vertex around it (and implying a Bauschinger effect, too). Such vertices around the loading point have been observed;²⁰ however, they appear to be a dynamic phenomenon only: they are not found after (even partial) unloading and reloading, for either definition of yield in aluminum (Fig. 5).

We shall focus our attention on back-extrapolated (macro)yield only. For this, 'isotropic hardening' is a good first approximation; deviations may be expected in two-phase materials (in which the Bauschinger effect has a more lasting component).

2.2.4. Plastic Anisotropy

It was already mentioned in Section 2.2.1 that experimental yield surfaces deviate from the allowed range of possibilities in isotropic materials; this is true even when the yield surface is determined at a given state of the material.¹⁸ Such discrepancies can only be explained on the basis of plastic anisotropy. There is also an anisotropy in the *strain* increments, which is in fact much more drastic: this corresponds to substantial deviations of the plastic potential from any possible isotropic yield surface. For example, if strips of rolled sheet are subjected to a uniaxial tensile stress, the two lateral strains will, in general, not be equal; their ratio is called the *R*-value or 'Lankford coefficient'.²¹ This anisotropy is a major cause of forming problems. Apart from the well known 'earring', it may be a cause of instabilities.²²

2.2.5. Conclusion

It is our judgment that, considering the 'realities' reviewed in Section 2.1, the concept of a yield surface is useful (in preference to a description in terms of a plastic modulus or a plastic viscosity)—but only for the back-extrapolation definition of yield. With this definition, 'isotropic hardening' may be a good first approximation, but plastic

isotropy is not. For the *rate* of hardening in different deformation modes, no good description is yet available.

3. FLOW KINETICS

Kinetics describes the rate at which processes occur under given driving forces (and given temperature)—or, conversely, the influence of an imposed rate of the process (such as straining), at a given temperature, on the forces required. The kinetics of a process is a consequence of the physical mechanisms that control it.

The physical mechanisms of plasticity occur on a microscopic scale; yet the kinetics is measured on a macroscopic scale. It is therefore important to have some understanding of the connection. Under certain conditions, the microscopic non-uniformities can give rise to macroscopic instabilities which in turn can lead to localization and thus return the process to a more microscopic scale. These questions will be briefly addressed in Section 3.1.

In plasticity, there are two different types of kinetics that are conveniently separated: *flow kinetics*, which describes dislocation glide at a given 'structure' or 'state' of the material; and *evolution kinetics*, which describes the influence of strain-rate and temperature on the rate of change of structure (state).

In Section 3.2, we will deal only with the kinetics of dislocation glide, and shall review only the most essential fundamentals. The statements that will be made may be regarded as well established, as a consequence of decades of research by many investigators. For a detailed treatment, see Kocks *et al.*¹

3.1. Non-uniform Deformation

3.1.1. Microscopic Heterogeneities and Jerkiness

Deformation is always non-uniform on a microscopic scale: it is localized in specific slip planes and, on a finer scale yet, occurs by the motion of dislocations. It is also non-uniform in a temporal sense: both dislocation motion and slip on a whole plane are 'jerky'. For most macroscopic applications, these effects are averaged out over the extent of the material element and over the time scale of interest.

This microscopic non-uniformity has one major consequence on constitutive descriptions of plasticity: the material element that is

taken to represent a 'point' in the continuum sense must be *large enough* to encompass many dislocations and many slip planes. This typically means that it must be larger than about $10\text{ }\mu\text{m}$ on edge. A more stringent criterion for most materials, which are polycrystalline aggregates, is that the material element must contain many grains. This typically means that it must be larger than about 1 mm on edge. On the other hand, of course, it must be *small enough* to warrant its treatment under macroscopically uniform conditions: the gradients of stress and of strain-rate, for example, should not be too great over the extent of the chosen material element.

Similarly, the time step over which deformation of a material element may be considered uniform is limited by various processes. Experimentally, the time for a 'unit slip step' (corresponding to the motion of many dislocations throughout the whole slip plane) is typically of the order of 0.1–1 s in observations at 'normal' strain rates and temperatures²³—but this may well depend on stress and temperature. It is limited at the lower end by dislocation vibrations in the phonon field, which occur on a time scale of the order of 10^{-10} s. This is a regime of interest in shock deformation, which has not been explored sufficiently.

One macroscopic consequence of the ubiquitous microscopic non-uniformities is that they provide a basis for (spatial or temporal) *fluctuations*—which may, on occasion, lead to *unstable* behavior.

3.1.2. Acceleration and Localization

Instabilities of deformation are usually described in terms of a localization of flow: that is, when flow becomes easier in one place than another (and the two are compatible with each other). 'Easier', at constant load, means faster; at constant rate, it means with a drop in load (and thus unloading of the 'other' elements). A characterization of plastic instability in terms of decreasing loads is, therefore, special and requires a knowledge of the interaction between many material elements and far-away boundary conditions.

On a local and instantaneous scale, the stress is fixed; however, the strain-rate may undergo fluctuations. The question then is whether these fluctuations tend to be damped out or lead to continued acceleration. In the latter case, the germs of instability are present.

There are two different types of causes for acceleration.³ One is in the evolution that would occur if the process went on: strain softening or texture softening, for example, leading to diffuse or localized

necking. The other possible cause of accelerations is instantaneous: e.g. if dislocation propagation is much easier than their generation; or if the propagation itself can occur in two modes—with or without continual aging. In both of these cases, the strain-rate may (and therefore will³) discontinuously jump to a higher value.

When there is a high generation stress for dislocations, such as in (aged) mild steel, there is a yield drop in tension (Fig. 2(a)), followed by a (perhaps jerky) stress plateau, while one or more Lüders bands propagate along the specimen. After the whole specimen has once been deformed, dislocation generation is no longer necessary and homogeneous deformation ensues.

When aging can occur dynamically during straining, there is a continual alternation between the two modes, leading to jerky flow (serrated stress/strain curves). This may occur essentially homogeneously, as in a compression test, or be accompanied by the rapid (non-steady) propagation of bands. It is associated with a macroscopically measurable negative rate sensitivity.

Neither of these processes is catastrophic, from a macroscopic point of view. The stress/strain/strain-rate relations are almost the same as under more uniform deformation. The main interest in the phenomena is in the surface irregularities that may be expected, for example in sheets that have deformed in a non-uniform manner.

From the point of view of constitutive relations, it is important to realize that macroscopic instabilities arise from within the regime of uniform behavior. One must therefore understand the plasticity of the material under presumed uniform conditions even if the primary interest were in the formation of instabilities.

3.2. Uniform Deformation

3.2.1. *The Mechanical Threshold*

The concept of a yield surface may be put as follows: it separates a region of equilibrium states, inside it, from a region outside it in which equilibrium is not possible. There is a very similar concept in dislocation theory, namely the 'mechanical threshold':¹ below this value of the stress, dislocations can always find a static equilibrium situation; above it, no such equilibrium configuration exists, at least in a statistical sense, for many slip planes. Note that some dislocation rearrangement may occur at very low stresses, and more and more as the stress increases: this is 'plasticity' in a sense, but it is akin to

'contained' plastic flow, the elastic-plastic transition region (Fig. 1(a)). The mechanical threshold corresponds exactly to the *asymptote* at which plasticity becomes general, macroscopic, as in Fig. 1(a). Thus, it is opportune to identify the 'yield surface' with the locus of mechanical thresholds.

The above considerations were strictly mechanical—to be exact: static. At stresses *above* the mechanical threshold, viscous and inertial forces come into play, and if these are included, a 'dynamic equilibrium' may be attained. On the other hand, *below* the mechanical threshold, thermal fluctuations (in 'thermodynamic equilibrium', at a constant, finite temperature) may release dislocations from their 'mechanical equilibrium' positions and give rise to 'thermally activated flow'.

Thus, the mechanical threshold serves as an important demarcation line: flow is in fact possible both below and above it, but the kinetics is quite different in the two regimes.^{1,24} This is shown schematically in Fig. 6. In a strictly rate-independent material, the strain rate $\dot{\gamma}$ would

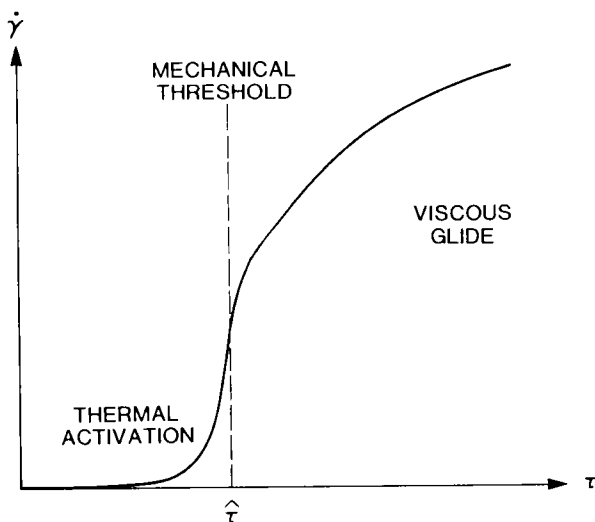


Fig. 6. Schematic linear strain-rate vs. stress diagram. The mechanical threshold $\hat{\tau}$ serves as a demarcation between two kinetic regimes: thermal activation (by far the most common process) below it; viscous glide above (presumably occurs at strain rates $>10^5 \text{ s}^{-1}$). The mechanical threshold would be the yield stress in completely rate-independent flow.^{1,9,24}

be zero at stresses below $\hat{\tau}$, and indeterminate at $\hat{\tau}$. The curve in Fig. 6 describes the effective smoothing-out of this step function in real materials (and also takes account of the upper limit set to all strain rates by the shear-wave velocity and the *de facto* maximum dislocation density).

The 'yield surface', if it is to refer to a rate-independent material, must then describe the mechanical threshold—and it will still be quite useful in rate-dependent materials. However, it is then no longer a limit between equilibrium (elasticity) and nonequilibrium (plasticity), but a limit between thermally activated plasticity (with little rate sensitivity) below this critical stress, and rate sensitive, quasi-viscous plasticity above.⁹ In most applications (very high strain rates excepted), *plasticity occurs inside this yield surface*—though close to it, because of the very low value of the typical rate sensitivity. There is usually no finite stress below which absolutely no irreversible flow occurs anywhere. (See Section 2.1.6.)

The mechanical threshold can be measured experimentally by determining the quasi-static yield stress under elimination of thermal activation, i.e. at the absolute zero of temperature. In practice, of course, this means by back-extrapolation to 0 K.† Note that rate effects *cannot* be eliminated by going to very small rates: this would drive the 'yield stress' essentially to zero.

3.2.2. Normalization by the Shear Modulus (Temperature, Pressure)

The mechanical threshold can, in principle, be calculated on the basis of dislocation theory (and a lot of statistics).¹ In almost all cases, it comes out proportional to some elastic constant (the exception being some precipitation hardened or ordered alloys, in which some interface energy may be the 'strength' factor). Thus, it is really an elastic strain that reaches a critical value at yield.

The elastic constants are functions of the temperature, and this gives rise to a 'trivial' temperature dependence of the yield stress. For this reason, the demarcation line between the two kinetic regimes should not actually be taken as the (absolute) mechanical threshold (the yield stress back-extrapolated to 0 K), but as this stress divided by the (temperature dependent) modulus.

† This back-extrapolation should be done from temperatures that exclude the region of very small temperatures, because of various extra mechanisms that occur in this regime,^{25,26} and it should be done on a plot of $\sigma^{1/2}$ vs. $T^{2/3}$.^{1,27}

All plastic processes are *shear* processes. Thus, it is some shear modulus that is important; the bulk modulus should not matter. The isotropic shear modulus is quite adequate for the purpose; in more exact treatments, some modulus connected with dislocation properties is more appropriate.¹ We shall use the symbol μ for whatever the appropriate shear modulus is. Similarly, we will generally characterize the mechanical threshold as a critical value of a *shear* stress, and label it $\hat{\tau}$. In this terminology, the important quantity is

$$\hat{\tau}/\mu(T, p) \quad (3.1)$$

Here, we have already incorporated another 'trivial' effect: the dependence of the shear modulus on the hydrostatic pressure p (the negative average normal stress). This is just one of the expressions of elastic non-linearity, and in very-high-strength materials this may be important (it is the major cause of the 'strength-differential effect').¹⁴ The hydrostatic pressure plays a special role inasmuch as it can reach values much higher than the shear stresses, when these are limited by plasticity.

For a rough estimate for close-packed metals, the following two relations are useful:

$$\left. \frac{\partial \mu/\mu_0}{\partial T/T_m} \right|_p \approx -0.5; \quad \left. \frac{\partial \mu}{\partial p} \right|_T \approx 2 \quad (3.2)$$

where μ_0 is the value of the shear modulus back-extrapolated to zero temperature, and T_m is the melting temperature.

3.2.3. Overstress, Dynamics

At stresses above the mechanical threshold, the rate of deformation is checked by viscous drag on dislocations (usually due to dislocation/phonon interactions). In this regime, it is useful to write the crystallographic shear rate $\dot{\gamma}$ in terms of the average dislocation velocity v and the mobile dislocation density ρ_m :

$$\dot{\gamma} = b\rho_m v \quad (3.3)$$

where b is the magnitude of the Burgers vector. Under drag control, v is proportional to the 'effective' (local) stress, which is zero at the mechanical threshold and increases above. All quantitative calculations show, however, that this 'effective stress' is *not* the difference between the applied stress and the mechanical threshold, but a strongly *nonlinear* function which asymptotes to the applied resolved

shear stress τ in the drag-controlled limit $\tau \gg \hat{\tau}$ (in practice $\tau > 2\hat{\tau}$ is ample).¹ Then

$$v = \tau b / B \quad (3.4)$$

The drag coefficient B has an important characteristic: it *increases* with temperature. (At high temperatures, it appears to reach a constant value according to experimental information, although it should be linear in T according to theory.¹) Thus, it is here not true (as it is for thermally activated glide) that higher temperatures lead to higher strain rates at the same stress.

The mobile dislocation density is a variable that is difficult to assess independently; however, rather general theoretical considerations suggest that it can depend on stress at most to the second power. Thus, for *viscous glide*:

$$\left. \frac{\partial \ln \tau}{\partial \ln \dot{\gamma}} \right|_T \geq 1/3 \quad (3.5)$$

It is very rate sensitive. It can be distinguished from other processes of high rate sensitivity (e.g. diffusional flow at high temperatures and low stresses) by the necessity that the applied stress be larger than the quasi-static flow stress at 0 K. In practice, this happens only at very high strain rates. (Follansbee²⁸ comes to the conclusion that this process has not as yet been observed in macroscopic deformation of polycrystals, in experiments up to about 10^5 s^{-1} .) We shall not be concerned with this regime in the remainder of this article.

Two other physical effects are important in some regimes of dislocation dynamics: relativistic behavior, for dislocation velocities in excess of, approximately, one-third the shear wave velocity; and dislocation inertia, in the regime where drag is small, at low temperatures (typically $< 20 \text{ K}$). Neither of these falls within our present 'interest space'.

3.2.4. Thermal Activation

By far the most common cause for rate sensitivity is a *lowering* of the flow stress (from the purely mechanical threshold) due to the help from thermal activation. The degree of lowering is the greater the higher the temperature, and the longer the time available for thermal activation (i.e. the lower the enforced strain rate). A quantitative link between temperature effects and rate effects is always given, in this

regime, by some form of Arrhenius law:

$$\dot{\gamma} = \dot{\gamma}_0 \exp(-\Delta G/kT) \quad (3.6)$$

involving a 'pre-exponential factor' $\dot{\gamma}_0$ and an 'activation energy' (more exactly, activation free enthalpy) ΔG . Both may depend on stress and also on the mechanical threshold. Thus, the rate dependence of the flow stress may be expressed (in its inverse) as

$$\frac{1}{m} \equiv \frac{\partial \ln \dot{\gamma}}{\partial \ln \tau} \bigg|_{T, \hat{\tau}} = \frac{\partial \ln \dot{\gamma}_0}{\partial \ln \tau} \bigg|_{T, \hat{\tau}} + \frac{1}{kT} \left(- \frac{\partial \Delta G}{\partial \ln \tau} \right)_{T, \hat{\tau}} \quad (3.7)$$

The first term is typically small (0, 1, or 2), while the second term is typically of order 100. Note that we have here specified the partial differentiation to be at constant mechanical threshold: this is a quantitative way of specifying 'constant structure' or 'constant state', as was done in words in the introduction of the rate sensitivity in eqn (2.1). (An observed rate sensitivity much greater than 0.01 almost certainly indicates an observation not at constant structure, but under control of evolution processes; see Section 5.2.4.)

The relation between activation energy and stress depends on the specific interaction profile between dislocation and obstacle, and on the way these are averaged over glide by many dislocations over many obstacles. It turns out that certain limiting considerations make the final relation rather insensitive to all these details. A sufficient approximation under most circumstances where temperature and strain-rate play a significant role, namely when short-range obstacles are rate-controlling, is¹

$$\Delta G = \mu b^3 g_0 [1 - (\tau/\hat{\tau})^p]^q = kT \ln(\dot{\gamma}_0/\dot{\gamma}) \quad (3.8)$$

where the exponents p and q are typically 1/2 and 3/2, respectively (generally, $0 < p < 1$ and $1 \leq q \leq 2$). The normalization with μb^3 is necessary when $\hat{\tau} \propto \mu$; g_0 is the total activation energy (if no stress were aiding) in units of μb^3 . We have also repeated eqn (3.6) in inverted form at the end of eqn (3.8); $\dot{\gamma}_0$ is typically 10^8 s^{-1} .¹

Figure 7 shows this behavior schematically (as a thin drawn-out line) for a value of g_0 that is far off-scale on the right. The heavy line indicates typical real behavior. (For many examples, see reference 29.) At very low temperatures (<50 K) dislocation-inertia effects may lower the flow stress slightly.^{25,26} At very high temperatures (typically above $T_m/2$), the flow stress is influenced by concurrent dynamic (and

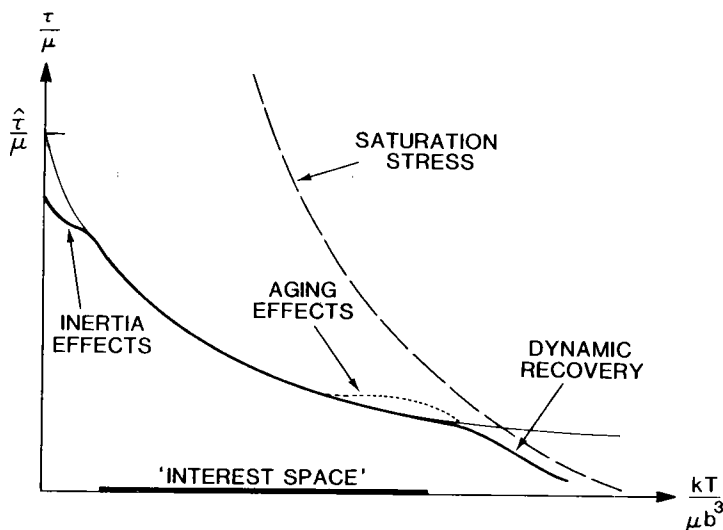


Fig. 7. Schematic temperature dependence of the yield stress (solid curve) and the (extrapolated) saturation stress (dashed), after appropriate normalization with the (temperature-dependent) shear modulus μ . Our 'interest space' covers typically the range of temperatures from 50 K to half the melting point.

eventually also static) recovery: the drop occurs near where the saturation stress (shown dashed) becomes lower than the extrapolated 'constant-structure flow-kinetics' curve. This regime is not well understood. Thus, the 'interest space' we have chosen is approximately as outlined by a heavy line in the abscissa. Temperatures above this are treated only under evolution.

At intermediate temperatures, there is often a 'plateau' in flow stress—at least apparently. This could have a number of reasons. First, it can merely be an effect of a large g_0 , as plotted. Often, however, it is assumed that there is an athermal contribution τ_0 to the flow stress; then, τ in eqn (3.8) would have to be replaced by $\tau - \tau_0$, and g_0 could be much smaller. The plateau itself is not sufficient reason for such an introduction of an additional parameter; we shall discuss methods to assess when this is warranted (Section 6.2.5).

When there are any solutes in the material, there are two further reasons for plateau-like behavior at intermediate temperatures. First, there should be a regime of dynamic strain-aging, which leads to the 'hump' above what would be the normal curve; this is indicated by a

dotted line in Fig. 7. Second, there are theories of solution hardening^{29,30} that predict a dependence of the activation energy on $1/\tau$ (times some short-range function as in eqn (3.8)): this gives a very slow decrease at higher temperatures.

Expressing the kinetic effects as influences on the flow stress, we have

$$\tau = \mu \cdot \frac{\hat{\tau}}{\mu_0} \cdot s \left(\frac{kT}{\mu b^3} \ln \frac{\dot{\gamma}_0}{\dot{\gamma}} \right) \quad (3.9)$$

(perhaps plus additional terms such as a τ_0). Here, s is the inverse of the function given in eqn (3.8).

While the semi-logarithmic dependence of the flow stress on the strain rate is thus founded in the physical mechanism of deformation, the relation is often expressed as a power law:

$$\tau = \tau_1 (\dot{\gamma}/\dot{\gamma}_1)^m \quad (3.10)$$

The exponent m is not identical to the rate sensitivity used in eqns (2.1) and (3.7): it is the same only when m is independent of τ ; in other words, when the activation energy ΔG is proportional to the logarithm of the stress. This is often sufficiently accurate over a substantial regime. Note, however, that m should be expected to depend on temperature if eqn (3.10) is merely a phenomenological approximation to eqn (3.6).¹

The quantities τ_1 and $\dot{\gamma}_1$ in eqn (3.10) are meant to be pairs of values under reference conditions within the regime of validity of eqn (3.10). They could, in principle, be replaced by $\hat{\tau}$ and $\dot{\gamma}_0$; but note that neither eqn (3.10) nor indeed eqn (3.6) holds near $\tau = \hat{\tau}$, so that $\dot{\gamma}_0$ is not a limiting value. In fact, in the neighborhood of the mechanical threshold, thermal activation and dislocation drag superpose in a complicated way.²⁷

3.2.5. Time Effects

In phenomenological plasticity, it is common to refer to 'time-independent' plasticity and, by implication, to possible 'time effects'. From the physical point of view, we have so far only referred to *rate* effects: these are in fact the most important ones.

However, there are some true *time* effects (which, of course, also reflect themselves dynamically as rate effects). They are of two kinds: recovery and aging. The first term is used for thermal softening (and may, in this context, also include recrystallization); it is important only

at temperatures above about $T_m/2$, at the upper end of our 'interest space', and we shall not consider it further here.

'Aging' refers to a change in the metallurgical structure with time: e.g. precipitation of solutes into small particles, or segregation of solutes to dislocations. It generally leads to a *hardening*—the more so the longer the time elapsed (or the higher the temperature: this is what led to the 'hump' in Fig. 7). When aging occurs dynamically, it leads to a *negative rate sensitivity*. While this is not truly a 'constant structure' effect, but rather a time evolution, the time scales involved may be so small that, for practical purposes, it acts like an 'instantaneous' rate sensitivity. Since this may lead to instabilities in deformation, it is a very important phenomenon. We will briefly discuss it in Section 3.2.

3.2.6. Stress Relaxation

The easiest test to measure kinetic effects is to merely stop the machine and watch the load relax. This relaxation is not usually due to a *time* effect, as discussed above (in fact, a *time-strengthening*, i.e. aging, effect can never be observed this way³), but is instead due to the decrease of strain-rate with stress. The plastic strain rate is proportional to the magnitude of the stress/time slope. Thus, a plot of this slope vs. stress gives a good first evaluation (more appropriate than stress vs. log t).

The low rate sensitivity of the flow stress means that the slope decreases very rapidly—until it is hard to distinguish from zero. This is another phenomenon that has tempted many observers to introduce an athermal, 'internal' stress level—although it can be easily explained without such a new parameter. (For further discussion of 'internal stresses', see Sections 3.2.7 and 6.)

At long times, the stress relaxation test is more difficult to do because it becomes very sensitive to temperature fluctuations; and it is more difficult to interpret because, at very low rates, simultaneous dynamic-recovery processes influence the results (much as they do the flow stress above $T_m/2$).^{2,12} On the other hand, if one is really interested in low rates, this is the quickest test.

The stress-relaxation test has other problems. The most important of these is that transients, which are so instructive for assessing material behavior, are very difficult to detect. Further, the small strains incurred do not guarantee that one is in the 'macroplastic' regime (Section 2.1.1); in fact, it is generally recognized that anelastic effects may play a significant role.³¹ Finally, if the stress-rate is ever

important (which actually seems rarely if ever the case^{32,33}), it has the opposite sign here.^{34,35} In conclusion, this author believes the stress-relaxation test to be grossly overrated in its usefulness and reliability. (See also Section 6.2.3.)

3.2.7. Strain Dependence

The most useful test of the flow-kinetics behavior of a material is to investigate the rate sensitivity as a function of strain. The best way to evaluate it is in the 'Haasen plot':³⁶ $m\tau$ ($=\Delta\tau/\Delta \ln \dot{\gamma}$) versus τ .³⁷ Note that this is not a relation between the 'activation volume' (which is proportional to $1/m\tau$) and the *applied stress at constant structure*: that would give information on the obstacle profile, which is not very sensitive (Section 3.2.4). Instead, it describes the evolution of the (inverse) activation volume with *structure*, as measured by the flow stress under standard conditions of strain-rate and temperature.

Figure 8 sketches various types of material behavior in a Haasen plot, as abstracted from extensive work on pure and commercial alloys:³⁷⁻⁴⁰ the data for pure materials (p) tend to back-extrapolate to the origin (this is called the Cottrell-Stokes law⁴¹); those for two-phase materials (t) back-extrapolate to zero rate sensitivity at a

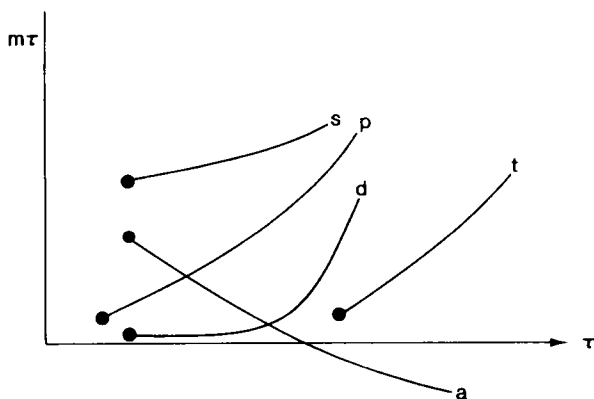


Fig. 8. Appropriate plot to diagnose the relative rate sensitivity of the flow stress, m , as a function of the flow stress itself (as both change with strain). Typical behavior, starting with the yield point, is shown for pure, solution hardened, and two-phase materials, as well as during dynamic strain-aging and under solute-dragging conditions. A negative rate sensitivity can cause instability.³⁸⁻⁴⁰

positive stress (this is *evidence* of the need for an additional 'internal stress' parameter); solution-strengthened alloys (s), on the other hand, have a relatively high rate sensitivity at the yield stress; dynamic strain-aging (a) causes a *decrease* of $m\tau$ with strain (and may eventually lead to a negative rate sensitivity and ensuing instabilities). Finally, while all curves tend to start out as straight lines (which is in itself evidence for a linear superposition of flow-stress mechanisms), they all tend to give an accelerated rate of increase of the rate-sensitivity when dynamic recovery becomes important at higher strains. An extreme case of this is solute dragging (d), which occurs at temperatures above a dynamic-strain-aging regime: here, the rate sensitivity increases rapidly at high strains, but is quite close to zero for a significant range of stresses.

In conclusion, even though the rate sensitivity is small, its variation with strain provides important clues as to which of many possible mechanisms in a particular material actually contributes significantly to the flow stress.

3.2.8. Conclusion

By far the most common cause of rate sensitivity at constant structure in uniform deformation under 'normal' conditions (our chosen interest space) is thermal activation. The rate sensitivity is small and serves to lower the flow stress from the mechanical threshold. The effects of strain rate and temperature are linked by the 'Fisher variable'⁴² $kT \ln(\dot{\gamma}_0/\dot{\gamma})$; if the flow stress vs. strain-rate relation is expressed, for phenomenological convenience, as a power law, the exponent is temperature dependent. The smallness of these rate effects makes the concept of a yield surface applicable and appropriate; the most precise correspondence is to the *mechanical threshold* of dislocation theory. This mechanical threshold is usually proportional to a shear modulus, and through it depends on temperature (uncoupled from strain rate in this instance) and on pressure.

While the 'instantaneous' rate sensitivity is so small that it may often be neglected, its dependence on strain provides a valuable tool for an analysis of important strengthening mechanisms.

4. POLYCRYSTAL PLASTICITY

Most materials are crystalline—but polycrystalline: they consist of many grains, each of which consists of an ordered array of atoms (on a

'lattice'). Typical grain sizes are between about $10\text{ }\mu\text{m}$ and 1 mm . (Grain sizes beyond these limits do occur, but we wish to exclude them from consideration here, because the large ones would usually not be small compared to the macroscopic body dimensions, and the small ones smaller than normal slip plane spacings.) The grain boundary thickness is of atomic dimensions; thus, the deformation of the polycrystal must be carried by the deformation of the grains.

The predominant mode of plastic deformation in single crystals is crystallographic slip; twinning plays a role at low temperatures, at very high strain rates, and in materials of low crystal symmetry (including, however, hexagonal lattice structures); and diffusive flow is important at high temperatures and very low stresses. Our 'interest space' was chosen such that slip is the controlling mechanism. The deformation of polycrystals then is some average over the crystallographic slips in each grain.

The deformations in the various grains cannot be independent of each other: certain conditions must be met at the interfaces between them, and this is in fact the major role of grain boundaries. The theory of polycrystal deformation consists essentially of treating these interactions—once the properties of the grains are known. The interaction may, however, influence the deformation mechanisms inside the grains themselves and make them different from single crystals. Thus, when we describe the plasticity of 'crystals' in the following sections, we mean that of *representative grains*.

4.1. Crystal Plasticity

4.1.1. Kinematics

Crystallographic slip has one very handy characteristic: it leaves the crystal lattice invariant (whereas elasticity changes both the spacings and the angles in the crystal lattice). Thus, our choice of omitting elasticity from the aspects of behavior we wish to treat (by concentrating on strains larger than, say, 0.2%) affords us a major bonus: an ingrown coordinate system. In fact, even if elasticity were added, it is an entirely appropriate way (see Sections 2.1.2 and 4.2.6) to treat plastic properties as those remaining after unloading (and to consider continuous deformation as a series of loadings and unloadings). Then, all the following relations are to be viewed as being in the unloaded state.

In general, the material properties are given in terms of the crystallographic axes (or in terms of axes of anisotropy for non-

random polycrystals) and must be related to the macroscopic coordinates for the solution of any particular boundary-value problem. This is the general meaning of 'orientation'. The material constitutive relations specify the relation between the 'true' (Cauchy) stress and the (true) strain rate, as well as the evolution with a (true) strain increment, *in these property-reference axes*.

Slip occurs on various 'slip systems': crystallographic planes and directions in which 'simple shear' takes place. If there is more than one such system (as always in grains of a polycrystal), the best coordinate system to choose is not usually one aligned with the slip elements, but one coincident with the crystallographic axes. We shall imply a cubic lattice here, and choose our coordinate system parallel to the cubic axes. (When the lattice coordinate system is not Cartesian, all relations are still valid, so long as one describes *plane* inclinations and spacings in the *reciprocal* coordinate system.)

If the incremental amount of simple ('engineering') shear in slip system s is called $d\gamma^s$, then the incremental (plastic) 'distortion tensor' (displacement gradient tensor) *in the crystal coordinate system* is given by

$$d\beta_{ij} \equiv \frac{\partial du_i}{\partial x_j} = l_{ij}^s d\gamma^s \quad (4.1)$$

where du is an incremental displacement of a point at location \mathbf{x} (in the crystal lattice) and l_{ij} is the tensor transformation matrix from the coordinates of the slip system to the 'crystal coordinates'. (If the components of the slip direction in the crystal system are b_i , and those of the slip plane normal are n_j , $l_{ij} = b_i n_j$.)

In eqn (4.1), we have introduced *incremental* quantities related to distortions and strains. These are meant to be infinitesimal, but not necessarily total differentials of (state) functions.^{43,44} Another common practice (more common presently) is to use *rates*. These can be obtained by dividing both sides of eqn (4.1)—and of all following equations that employ the differential symbol d —by dt , again without implying that deformation 'rates' are total derivatives of any quantity that measures finite strain. In this article, we use increments in order to keep clear of the implication that directions in strain space have anything to do with the physical speed with which a process occurs (a *scalar* rate). Either increments or rates are appropriate in the spirit of state-plus-evolution descriptions; it is imperative that history variables such as *any* measure of finite strain be avoided.

The change in all material lines with respect to the crystal lattice is fully described by eqn (4.1): both in their length as in their orientation with respect to the crystal lattice (and, by implication, their change in angle with respect to each other). Since the element does not generally remain cubic in shape, a specification of its 'orientation change' (or its 'rotation' with respect to the lattice) requires further thought.⁵¹ What is actually meant by the orientation change of a grain is the change in the relative orientation of the lattice and *certain material lines and/or planes*. In simple cases, these are the plane and line kept constant in the macroscopic experiment. For example, in rolling, the rolling plane and the rolling direction are kept constant (and are the coordinates in which the stress and strain state are related to the boundary conditions); but, for example, a plane that was initially perpendicular to the rolling direction need not remain so.

If the incremental displacement gradient tensor in *machine coordinates* is labelled dB_{ij} , then when a *plane* i remains parallel to itself, $dB_{ij} = 0$ ($j \neq i$); and when a *material line* j remains parallel to itself, $dB_{ij} = 0$ ($i \neq j$). The components of the ('relevant') *orientation change* $d\Omega_{ij}$ are given by the values of $d\beta_{ij}$ for those three components in which $dB_{ij} = 0$. Thus, the orientation change is *not*, in general, the antisymmetric part of the distortion tensor $d\beta_{ij}$.⁵¹

The incremental strain, on the other hand, is always the symmetric part of the incremental distortion tensor. We write

$$d\epsilon_{ij} = m_{ij}^s d\gamma^s, \quad m_{ij}^s \equiv \frac{1}{2}(l_{ij}^s + l_{ji}^s) \quad (4.2)$$

The 'Schmid factor' m_{ij} plays the role of a 'strain direction': a unit 'vector' (Section 2.2.1) in strain-increment space for each slip system.

4.1.2. Work Increment and 'Resolved' Stress

Consider now the increment of plastic work per unit volume:†

$$dW = \sigma_{ij} d\epsilon_{ij} = \sigma_{ij} m_{ij}^s d\gamma^s \quad (4.3)$$

where σ is the true stress applied to the volume element under consideration. Use of eqn (4.2) in the second part of this equation has produced a scalar

$$\sigma^s \equiv m_{ij}^s \sigma_{ij} \quad (4.4)$$

† Summation over repeated (upper or lower) indices is always implied. When pairs of indices are repeated, a sum over all *pairs* is meant.

It is the value of the *resolved stress* in system s , or the projection of the stress vector on the direction of the strain-increment vector, or the work-conjugate to the crystallographically defined scalar strain-increment measure $d\gamma$ in that system.

The 'Schmid law' states that this resolved stress must reach a critical value for system s to operate.⁵³ Then, the yield condition is

$$\sigma^s(\sigma_{ij}) = \tau^s(T, p, \dot{\gamma}^s, \dots) \quad (4.5)$$

Note the essential difference between σ^s and τ^s (even though both are shear stresses; see eqn (4.4)). This difference is emphasized by the functional dependences written explicitly into eqn (4.5): σ^s is a function of the applied stresses (representing the weights on the body; see Section 1); τ^s is a material property, a special case of the 'plastic resistance'.⁴⁵ Ideally, it should be the mechanical threshold $\hat{\tau}$ (the rate-independent value of τ^s at $T = 0$ K), but eqn (4.5) allows for the more realistic case of a flow stress at finite temperature—which is lower because of $\mu(T)$ and, in addition, because of thermal activation. For the latter reason, the strain rate $\dot{\gamma}^s$ on system s is also relevant. The pressure dependence of τ^s is a stress dependence of a different nature from that of σ^s : it is an effect on the state of the material, for example because of $\mu(p)$, or because of a change in the dislocation core structure with pressure. In principle, τ^s may even depend on deviatoric components of the applied stress.

We shall always use σ^s for applied stresses (or functions thereof) and τ^s for material properties. The difference is crucial; if the particular terminology is not desired, two different symbols must in any case be found for 'stress' and for 'strength'.[†] (For consistency, the yield strength should not have been called σ_Y in Fig. 1, but τ_Y , with some orientation factor.)

4.1.3. The Plastic Potential

Since the scalar stress measure σ^s was chosen as the work-conjugate of the scalar strain-increment measure $d\gamma^s$ for single slip (eqns (4.2) and (4.4)), it follows immediately that[‡]

$$d\varepsilon_{ij} = \frac{\partial \sigma^s}{\partial \sigma_{ij}} d\gamma^s \quad (\text{no sum on } s) \quad (4.6)$$

[†] The new DIN norm uses R for the plastic resistance.

[‡] The equation does hold for multiple slip when it is summed over s , but this is here not intended.

In words, if σ^s is known as a function of σ_{ij} , it acts like a plastic potential for single slip: the direction of the plastic strain increment is parallel to the gradient of σ^s ; it is *normal* to a surface

$$\sigma^s(\sigma_{ij}) = \text{constant} \quad (4.7)$$

where constant means independent of stress. In the *yield condition* (4.5), the right-hand side was τ^s —which may or may not be independent of stress; we have seen that τ^s does, in principle, depend on pressure, at least through the shear modulus. '*Normality*' (*of the strain increment on the yield surface*) holds when the material strength parameter is insensitive to stress. This is generally true for metals in the regime of interest here, but it is, for example, grossly untrue in polymers.¹⁵

It should also be mentioned that normality may be violated for another reason: namely, when some strain increment component may be indirectly caused by plastic deformation; for example, a change in volume because of vacancy or dislocation generation. This is also usually negligible (see Section 2.1.5).

In summary, the 'resolved stress', i.e. the scalar stress measure that is work-conjugated to the scalar plastic strain-increment, is the plastic potential (for which normality is true by definition). Normality of the strain-increment on the *yield surface* is not a general principle, but may or may not hold depending on the class of material and the range of the imposed conditions. It generally holds for the 'interest space' we have defined in Section 1, and we will thus be lax in keeping plastic potential and yield surface separate.

4.1.4. The Yield Surface of Single Crystals⁵³

Descriptions in stress space are most convenient in 'vector' (5×1 matrix) notation (Section 2.2.1). With some fixed assignment of the single (Greek) subscripts to the five independent pairs of double subscripts, one can then write eqns (4.4) and (4.5) as

$$m_v^s \sigma_v = \tau^s \quad (4.8)$$

This is a set of planes (if τ^s is independent of the stress component σ^s),[†] one plane for each slip system s . The yield condition specifies not only that eqn (4.8) must be fulfilled for every active system, but also

[†] When τ^s depends on the stress state, the yield surface may even be concave; the plastic potential is convex by definition.

that $\sigma^s \leq \tau^s$ for every other system. Thus, the yield surface is the *inner envelope* of the planes specified by eqn (4.8). Each of these planes has intercepts $\tau^{(s)}/m_v^{(s)}$ on the v -axis.

Figure 9 illustrates the typical topology for three slip systems (a, b, c). The inner-envelope construction means that systems 'a' and 'c' cannot operate simultaneously under any stress state. Whenever more than one system is active, the stress vector must be at a 'vertex' in a space the dimension of which equals the number of *independent* systems operative. Some slip system combinations are not independent; imagine the vertex of an octahedron in three dimensions: four planes meet, but only three (non-coplanar) strain directions are necessary to compose an arbitrary strain vector.

Vertices in the single-crystal yield surface play a major role in polycrystal plasticity. It is convenient to define normalized vertex vectors M_v^s such that the stress for a particular vertex v is

$$\sigma_v^s \equiv M_v^s \tau \quad (4.9)$$

where τ is a single scaling parameter for the yield surface. To fulfill the yield condition, we must have (eqn (4.8))

$$M_v^s m_v^s = \tau^s / \tau \quad (4.10)$$

for each active slip system s in the vertex v . The right-hand side is equal to 1 when all slip systems have the same critical resolved shear stress (and then the left-hand side is 0 for all non-active slip systems in the case of fcc crystals).

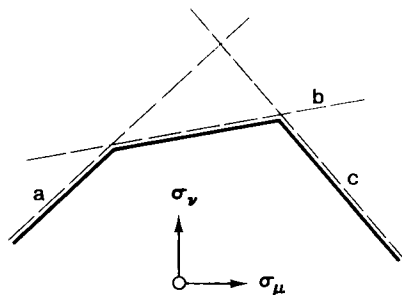


Fig. 9. Schematic single-crystal yield surface: three facets for the slip systems a, b, and c (each in *one* of their possible signs). To activate two systems simultaneously, the stress must be at a vertex; some systems (such as a and c) cannot be activated simultaneously.⁵⁷

Another relevant quantity is the work-conjugate of τ ; in the special case outlined above, it becomes the algebraic sum of shears:

$$d\Gamma \equiv dW/\tau = M_v^\gamma d\epsilon_v = M_v^\gamma m_v^\gamma d\gamma^s \quad (4.11)$$

Observe the following duality. A stress vector of arbitrary direction will generally produce slip on a single system (it will touch a single facet of the yield surface); thus, it will produce one of a few *discrete* directions of *straining*. This is, for example, seen in free single crystals under tension. On the other hand, if one imagines the direction of the strain-increment vector as given, the *stress* must be one of a small number of *discrete* values: the vertex vectors. It is evident from the plastic potential concept that the prescribed strain direction must be contained in the cone of normals of the active vertex.

There are two dual kinds of non-uniqueness associated with the facets-plus-vertices nature of the single crystal yield surface: if one happened to prescribe a strain direction that is identical to one of the facet normals, the stress direction is not unique; and if one happened to prescribe a stress vector that points precisely into a vertex, the resulting strain-increment direction is not unique.

Finally, there is an ambiguity at those vertices in which more facets meet than is the 'order' (dimensionality) of the vertex: then, a given strain-increment vector cannot uniquely be decomposed into shears on the activated slip systems. This has been a topic of much discussion in the theory of polycrystal plasticity, since the strain in the grains is presumed completely prescribed (in 5 dimensions) in the most prominent model,⁴⁶ whereas the vertices in highly symmetric materials always activate more than 5 slip systems.

4.1.5. Rate Sensitivity

As we have pointed out above (Section 3.6), the most precise definition of the yield surface is as the (rate-independent) mechanical threshold; but the typical rate sensitivities are so low that a 'flow surface' can be defined for a constant strain rate, which differs only very little from the limit yield surface.⁹

Figure 6 showed (exaggerated) the typical rate effect: a slight lowering of the flow stress from the mechanical threshold. Thus, for each single slip system, a constant shear rate $\dot{\gamma}$ will correspond to a plane parallel to the rate-independent yield facet, slightly inside it (Fig. 10).

When two or more slip systems operate, however, each one can

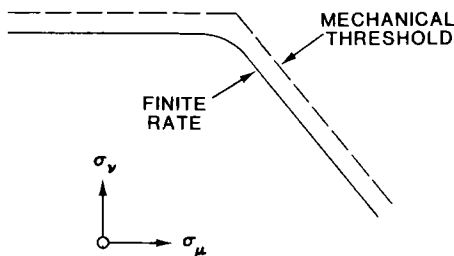


Fig. 10. The mechanical threshold provides a 'true' yield surface, in the rate-independent limit. Thermal activation lowers the stress needed for a given strain rate. Where two slip systems operate simultaneously, the yield surface vertices become rounded.⁹ The relative decrease in stress is proportional to the relative rate sensitivity, typically of order 0.01.⁴⁸

operate at a lower shear rate to produce the same macroscopic strain rate, and thus at an even slightly lower stress. Thus, *the 'flow surface' near a vertex gets rounded; this eliminates all problems of non-uniqueness.* Note that, in order to achieve this very convenient result, the rate sensitivity need not be large; it only must exist in principle (and be positive definite). Thus, the concept of an *essentially* rate-independent (a 'rate-insensitive') plasticity, but not a strictly rate-independent one, allows one to use the methods of classical plasticity theory, including (sharp) yield surfaces, while avoiding some of its pitfalls.^{47,48}

When the rate sensitivity is low, the (true) vertices become sharp 'noses' (Fig. 10). These still have the characteristic that a very small variation in stress direction can produce a very large variation in strain-increment direction. This property of a vertex is important in the consideration of instabilities. The change from a true vertex to a sharp nose eliminates the possibility of bifurcation; but it leaves intact the sensitivity to small fluctuations, which may (or may not, depending on other factors) lead to instability.⁴⁹

We shall, in the following, use the term 'vertex' to signify a sharp nose, whether it be continuous or discontinuous.

4.2. Averaging over a Polycrystal

4.2.1. Nomenclature

We have already introduced the vector notation for stress and strain-increment, which we write in extended form with single (Greek)

subscripts that go from 1 to 5. The particular association of one Greek subscript with one pair (or an independent combination) of Latin ones is arbitrary; different ones have been used. Two conditions should, however, be met: that the stress vector and strain-increment vector be work-conjugated, so that

$$dW = \sigma_v d\epsilon_v \quad (4.12)$$

and that, if the coordinate system is not Cartesian, the strain-increment vector be measured in reciprocal space, i.e. by the inverse axis intercepts of a plane perpendicular to it.

The relation between stress and strain-increment on the yield surface does not involve the magnitude of the strain increment, only its direction. It is convenient to define such a 'magnitude' and 'direction':⁵⁵

$$d\epsilon_v = \epsilon_v^0 d\epsilon \quad (4.13)$$

The straining direction ϵ^0 should be normalized in some fashion. It has become common to do this such that, in uniaxial tension, the scalar $d\epsilon$ is equal to the measured increment in tensile strain; then

$$\epsilon_v^0 \epsilon_v^0 = 3/2 \quad (4.14)$$

Inserting eqn (4.13) into eqn (4.12) leads to the definition of a scalar stress measure:

$$\sigma \equiv \sigma_v \epsilon_v^0 \quad (4.15)$$

which, when multiplied by the scalar $d\epsilon$, again gives the work. It is equivalent to the *resolved stress* in single crystals, and we will retain this descriptive nomenclature for the general case. σ is the distance in stress space from the origin to a plane perpendicular to ϵ^0 that is tangent to the plastic potential surface. In fact, the scalar σ is the plastic potential, much as the resolved stress on a slip system was shown to be the plastic potential for single slip (Section 4.1.3). If the resolved stress is *constant*, you get the von Mises sphere in deviatoric stress space; but, in general, the distance σ may depend on the direction ϵ^0 .

One may also define a *stress direction* σ^0 so that

$$\sigma_v \equiv \sigma_v^0 \sigma \quad (4.16)$$

Note that σ^0 is not normalized to its length, but to its projection on ϵ^0 (eqn (4.15)). If one gave σ as a function $\sigma(\sigma^0)$, it would be a plastic potential in the conventional sense.

For *vertex* stress vectors, one may finally write, with eqns (4.10) and (4.15):

$$\sigma = M_v \varepsilon_v^0 \tau \equiv M \tau \quad (4.17)$$

where a scalar M , the *normalized Taylor factor*, has been defined as the projection of a vertex vector on the prescribed normalized strain direction.⁵⁵ (Note that this normalized definition differs from the standard one for deformation modes other than uniaxial tension or compression.) M is the spacing of tangent planes of the yield surface normalized to the scalar τ . The stress direction becomes

$$\sigma_v^0 = M_v / M \quad (4.18)$$

The 'maximum work' postulate by Bishop and Hill⁵⁷ gives a prescription for the derivation of the correct value of M for any given strain direction: it is the greatest of all those for any vertex on the yield surface; this guarantees that the prescribed strain direction is contained in the cone of normals of the activated vertex.

4.2.2. The Model

A grain in a polycrystal finds itself under a number of boundary conditions on stress and strain-increment. (In this section, we address total, not deviatoric, stresses and use the symbol σ for these.) A useful idealization is that some strain-increment components are prescribed and some (the complementary) stress components are prescribed. If we mark the prescribed components by an overwritten double bar, and furthermore assume that any prescribed component is equal to the average (single bar), we have

$$\begin{aligned} d\bar{\bar{\varepsilon}}_v &= d\bar{\varepsilon}_v & (p \text{ values of } v) \\ \bar{\bar{\sigma}}_\kappa &= \bar{\sigma}_\kappa & (6 - p \text{ values of } \kappa \neq v) \end{aligned} \quad (4.19)$$

where any prescribed stress component is usually zero.

There is an easy case in which eqns (4.19) in fact give the complete (approximate) solution: that of plate-like grains, with large planar interfaces (the interactions across the other interfaces being neglected). Across a planar interface, there are three continuity conditions on strain and three (the complementary ones) on stress. Thus, both compatibility and equilibrium can be fulfilled.⁵⁰⁻⁵²

Another easy case is the upper bound solution: all plastic strain increments are considered prescribed ($p = 5$); the one remaining stress

condition is on the hydrostatic stress (which is here conveniently set to zero). In this case, compatibility is fulfilled *a fortiori*, but one renounces on fulfilling equilibrium. This upper bound has been considered a good approximation for large-strain plasticity. A justification can be given as follows:⁵³ The internal stress distribution necessary to fix up the violations of local equilibrium will disturb the compatibility conditions only by elastic strains, small compared to the plastic ones. On the other hand, violations of compatibility that are being fixed up elastically would add enormous stresses. Thus, a sensible approach to approximate polycrystal plasticity theory is: '*compatibility first*'. This is the essence of Taylor's model,⁴⁶ which is used as a basis for most current simulations of polycrystal plasticity.^{47,54-56}

Compared to this completely constrained upper-bound solution, the more general approach of eqns (4.19) has been called 'relaxed constraints' (RC): only the most important compatibility conditions for a given grain shape are fulfilled by prescribing the respective strain components to be uniform; equilibrium is fulfilled for the remaining components.^{52,55}

4.2.3. The Calculation⁵⁵

For each grain, one must determine which stress components can be prescribed (to be zero) and then investigate the yield surface in the subspace $\sigma_\kappa = 0$ for $6 - p$ values of κ . This subspace is p -dimensional. The yield subsurface has vertices of order p , with vertex vectors $\sigma^v = \mathbf{M}^v \tau$. Among them, the right one (components σ_v) for the prescribed strain-increment in this subspace (components $d\bar{\epsilon}_v$) follows from the geometric criterion

$$(\sigma_v - \sigma_v^v) d\bar{\epsilon}_v \geq 0 \quad (p \text{ values of } v)$$

or, normalizing both the stress and the strain increment:

$$(M_v - M_v^v) \bar{\epsilon}_v^0 \geq 0 \quad (4.20)$$

This is an extension of Bishop and Hill's⁵⁷ maximum work hypothesis to relaxed constraints, proved by Renouard and Wintenberger.⁵⁸

As a result of the calculation, the unknown stress components σ_v of the correct vertex fulfill the conditions

$$\sigma_v \bar{\epsilon}_v^0 = M \tau \quad (4.21)$$

and, according to the model (or at least as an upper limit), the

polycrystal stresses must fulfill the same condition for the averages:

$$\bar{\sigma} \equiv \bar{\sigma}_v \bar{\epsilon}_v^0 = \bar{M} \bar{\tau} \quad (4.22)$$

In truth, the right-hand side should be averaged as a product; however, it has been found that the error is small. (Alternatively, one may interpret the quantity $\bar{\tau}$ as a *weighted* mean, weighted by M ; this has been called $\bar{\tau}$.⁵⁵)

Equation (4.22) is, again, for arbitrary stresses, an equation of a plane perpendicular to the direction $\bar{\epsilon}^0$, at a spacing $\bar{M} \bar{\tau}$ from the origin. \bar{M} depends on the crystallographic orientation distribution function of the grains in the polycrystal, and on the direction of the prescribed strain increment $\bar{\epsilon}^0$; $\bar{\tau}$ depends on the substructural state of each grain. The inner envelope of all these planes constitutes the yield subsurface for the polycrystal in this space.

While this 'inner-envelope construction' of the (upper-bound) yield surface follows directly from the maximum-work principle (4.20), it is sometimes more convenient (and will be needed later here) to know the locus of the stress vector itself. In fact, Hutchinson⁵⁹ has described the polycrystal yield surface by

$$\bar{\sigma}_v = \bar{M}_v \bar{\tau} \quad (4.23)$$

(again with the proviso that $\bar{\tau}$ is a weighted mean). This can be shown to hold under at least one of the following two assumptions: (a) equilibrium everywhere, which would seem to be inconsistent with an upper-bound model; or (b) normality assumed for the polycrystal independently from the single crystal. (Normality for the polycrystal can be derived from that of the single crystal only by assuming equilibrium.) As we have shown, normality does not need to hold (and in fact does not hold when τ depends on the stress state). However, for the plastic potential, normality holds by definition—for the polycrystal too. Thus, we will make use of relation (4.23) *without* the factor $\bar{\tau}$: *the plastic potential of the polycrystal is the five-dimensional locus of the vector $\bar{\mathbf{M}}$.*

The strain components in the *unprescribed* directions (where the stress is zero) follow, with eqn (4.2), from the crystallographic shears, which are a solution to the p equations (see eqn (4.19))

$$d\epsilon_v = m_v^s d\gamma^s \quad (4.24)$$

(modified near vertices at which the number of slip systems s exceeds the number of prescribed components λ according to the rate-

sensitivity criterion; Section 4.1.5). These crystallographic shears also determine the orientation change, according to eqn (4.1).

The average algebraic sum of shears (or, more generally, the work conjugate of τ) follows easily, as in eqn (4.11) and using eqn (4.17):

$$d\bar{\Gamma} = \bar{M} d\bar{\epsilon} \quad (4.25)$$

In some theories of hardening, $d\bar{\Gamma}$ plays a physical role;⁴⁶ but even when it does not it can serve as a handy 'microscopic equivalent strain increment'.⁶⁰

4.2.4. The Yield Surface of Polycrystals

The method described above has been used to calculate polycrystal yield surfaces after various simulated deformations⁶¹ (and also to calculate stress/strain curves for various proportional tests⁵⁵). Figure 11(a) shows one of the most interesting applications: the yield surface after a torsional shear of 1.0, in the subspace that also displays the normal stress and strain-increment along the torsion axis. It is seen that, if the longitudinal stress is kept to zero ('free ends'), there results a component of the strain-increment direction in the 'lengthening' direction: this, we feel, is the best explanation of this well known effect so far. Conversely, when the ends are fixed (which is the boundary condition under which the prestrain was run), an axial compressive stress develops.

The two effects are shown quantitatively in Fig. 11(b). The surprising result is that they are not proportional to each other—and this is actually observed.⁶² Thus, they are not just two ways of expressing the same thing. The yield surface in Fig. 11(a) may help in understanding this phenomenon.⁶¹ The length change of a 'free' tube is proportional to the inclination of the yield surface at the loading point; but the compressive stress in fixed-end torsion depends on the location of the apex. The two are not necessarily related (except in sign); the shape of the yield surface is quite irregular.

The irregularity of the yield surface is a very general observation. Figure 12(a) shows one after simulated rolling to a true through-thickness strain of 3.0, in a space of two shear stresses. In an isotropic material, the yield surface would have to be a circle in this space, regardless of any assumptions (von Mises or otherwise). In classical treatments of anisotropy, it would be an ellipse⁶³ or an oval of some kind.^{64,65} It is neither, it has vertices. At this strain, the texture is not

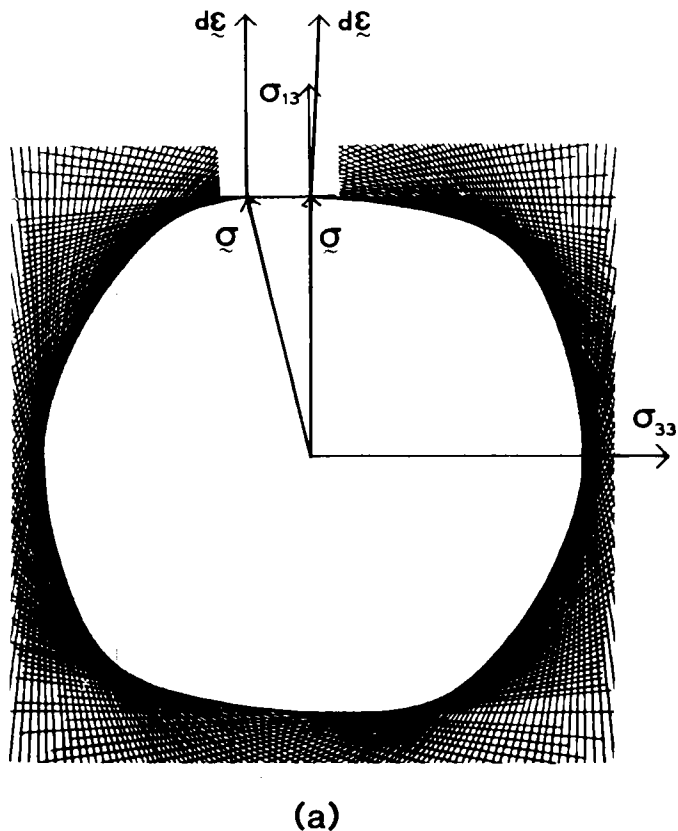


Fig. 11. (a) A polycrystal yield surface derived by computer simulation,⁶¹ after torsion to $\gamma_{31} = 1$, with fixed ends. Continued shear with fixed ends demands a compressive stress (at the apex to the left); continued shear with free ends would give the length change corresponding to the inclination of the normal at the top.

yet very sharp—not at all like a single crystal (for which one would expect vertices).

Finally, Fig. 12(b) shows the same yield surface, but now in the ‘ π -plane’: where the three interdependent normal stresses make a tristar. Note that the normal stresses are here meant to lie in the directions of the (orthotropic) texture symmetry, *not* necessarily in the

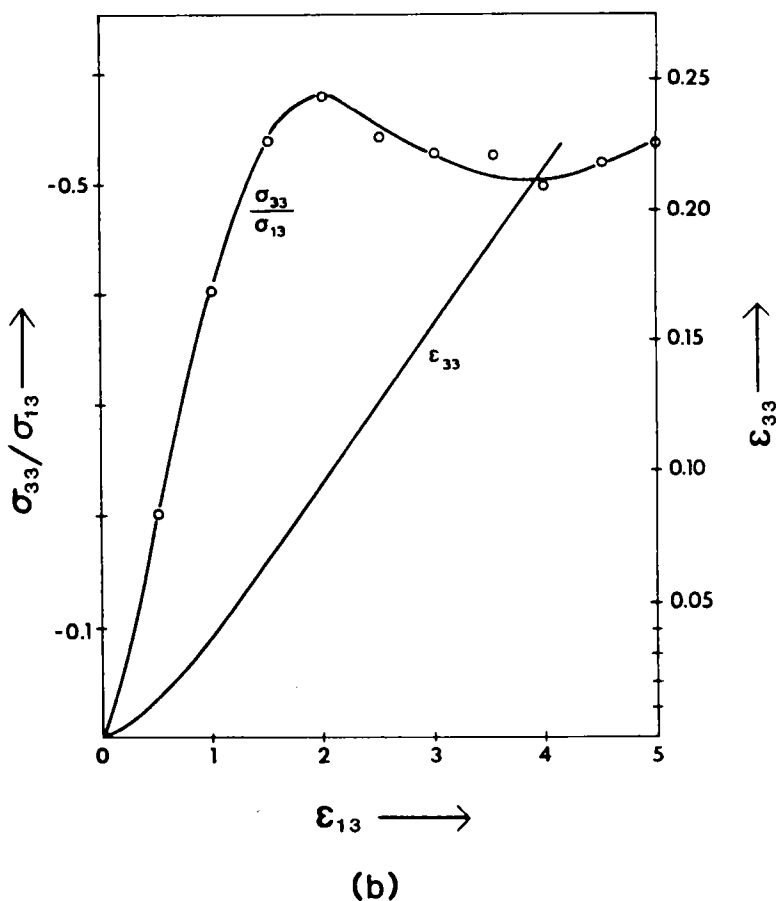


Fig. 11. *contd.* (b) Experimental results⁶² for both cases.

principal axes. The most obvious feature of this yield surface is the sharpness of its vertices. Note also that the long flat region perpendicular to the previous rolling direction is the *edge* that corresponds to the vertex in the direction of the abscissa of Fig. 12(a).

The yield surfaces shown here were derived on the basis of a computer code at various stages of its development: they are not to be taken as quantitatively reliable results. However, *all* polycrystal yield surfaces we have ever derived, after strains in excess of about 50%

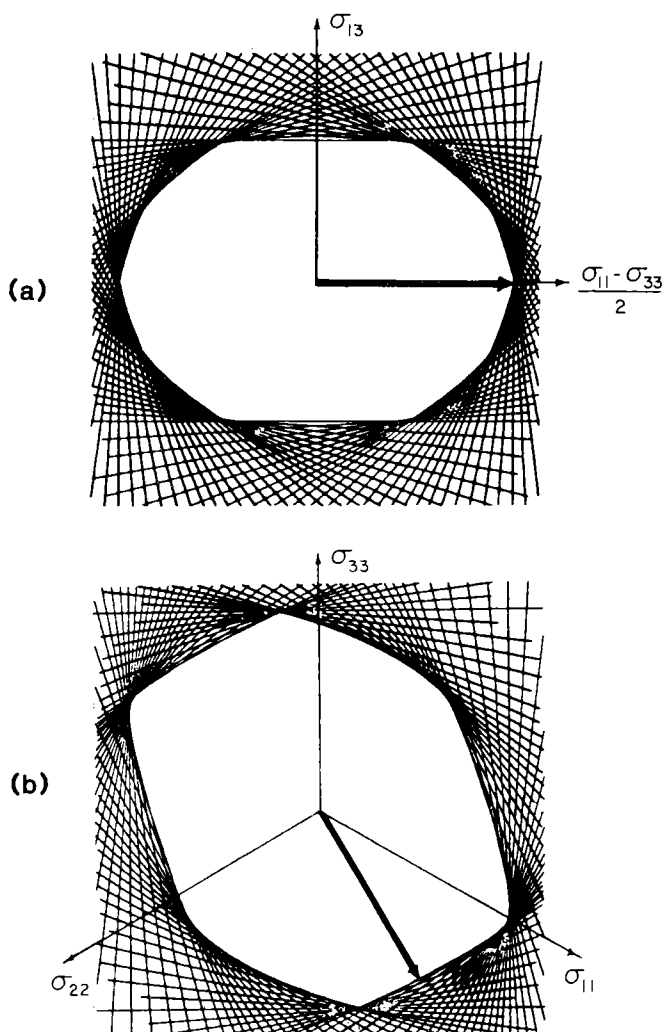


Fig. 12. Calculated polycrystal yield surface after plane-strain rolling to a thickness reduction of 95%. An isotropic yield surface would have to be circular in the shear space (a) and would be circular under the von Mises assumption in the normal-stress-deviator plane (b). The calculated yield surface shows vertices and flat spots.

from an isotropic start, had vertices, ridges, and flat spots. (It appears that they are particularly prominent under 'relaxed-constraints' (flat-grain) conditions.) Vertices are very hard to observe experimentally; nevertheless, some indications have been seen. More clearly, 'flat' regions have been identified in some cases.⁶⁶

The most important consequence of vertices is that they may lead to instabilities.⁶⁷ This has been treated in some detail both for the ridge perpendicular to the rolling direction (using the plane-strain idealization for the rolling process), which can be one cause of 'shear bands',⁶⁸ and for localized necking in sheets, which can be explained on the basis of the sharp vertex in Fig. 12(b).

As in the case of single-crystal plasticity, rate dependence will round off any polycrystal vertices. However, the corner would (because of the quantitatively low rate sensitivity) remain very sharp. Then, minute internal stresses could lead, in practice, to the same kind of non-uniqueness that true vertices represent.

All the yield surfaces shown so far have been projections into a particular two-dimensional space. The most general yield surface would have to be plotted in five dimensions. However, in two interesting cases, three dimensions are actually sufficient (and only one octant of three-dimensional space, when there is no sign dependence of yield). One of these is a *fiber* texture, the other an orthotropic texture (such as in a rolled sheet) when only *plane-stress* conditions are encountered (such as in sheets). In both cases, these subspaces are 'closed': an arbitrary stress in this space gives rise to strain-increment components *only* in this space, and vice versa. The figures are simultaneously sections and projections of the five-dimensional yield surface.⁶¹ Such symmetry relations can be of crucial help in treating anisotropic materials.

4.2.5. Constitutive Relations for Anisotropic Plasticity

The rapid development of extensive flat spots and rather sharp ridges and vertices on the plastic-potential surface of polycrystals during deformation makes a description in terms of *functions* difficult if not impossible. Perhaps some scheme will eventually be found that abstracts the essential features of realistic yield surfaces in a useful way. For now, it would seem that one must employ a *tabular approach*. What is needed is a table of the stress direction σ^0 for each strain-increment direction ϵ^0 . Unfortunately, this table will change with strain, since the *shape* of the yield surface changes. A help is that

the symmetries of the material (if known) and of the test to be performed can substantially reduce the space in which ϵ^0 must be scanned (Section 4.2.4).

The relation between σ^0 and ϵ^0 may be expressed in matrix form:

$$\sigma_\mu^0 = P_{\mu\nu} \epsilon_\nu^0 \quad (4.26)$$

Note, however, that P is in general different for each direction ϵ^0 ; it does not transform like a fourth-rank tensor as Hart assumed.³¹ Also note that it is substantially different from a 'plastic modulus': it links strain-increments (or rates) with (total) stresses. This is because we elected to treat plasticity only, and in terms of a flow-rule-type theory, not a modification of an elastic one. Dimensionally, it is like a viscosity; but it has nothing to do with kinetics.

A table of the 5×5 matrix \mathbf{P} for each vector ϵ^0 may sound more cumbersome than a table of the vector σ^0 vs. ϵ^0 . However, under many symmetry conditions, \mathbf{P} may have simple properties. For example, in *isotropy*

$$P_{\mu\nu} = \delta_{\mu\nu} \quad (4.27)$$

in any subspace defined by shear axes only (including differences between diagonal components), as a consequence of isotropy itself, and further in the deviatoric principal plane *under the assumption of a von Mises potential*. In anisotropic materials, eqn (4.27) may hold exactly for some subspaces, and may be an acceptable approximation for some other regions. Conceivably, one needs to explore the real nature of \mathbf{P} only near extensive flat spots or sharp vertices that may be reached in the expected application.

Given the table $\mathbf{P}(\epsilon^0)$ and, as we shall need, $\bar{M}(\epsilon^0)$, we now imagine a polycrystalline material element under strain-increment boundary conditions in a p -dimensional subspace:

$$d\bar{\epsilon}_\nu = \bar{\epsilon}_\nu^0 d\bar{\epsilon} \quad (p \text{ values of } \nu) \quad (4.28)$$

where the double-bar, as before, means 'prescribed'. The average deviatoric stress in this subspace follows from

$$\bar{\sigma}'_\mu = P_{\mu\nu} \bar{\epsilon}_\nu^0 \bar{M} \bar{\tau} \quad (p \text{ values of } \mu) \quad (4.29)$$

The evolution of \mathbf{P} and \bar{M} must be calculated or measured and is to be tabulated. That of $\bar{\tau}$ follows from the strain-hardening rate $\theta \equiv d\tau/d\Gamma$ to be discussed in the next section, where

$$d\Gamma = M d\bar{\epsilon} \quad (4.30)$$

We have implied above that the subspace in which the strain-increments are prescribed may have $p < 5$ dimensions. This applies only when a relaxed-constraints theory is appropriate. For example, in a cold-rolled sheet, all grains are likely to be flat and parallel to the sheet; then, the normal stress in the thickness-direction may be relaxed on the individual grains and prescribed (to be zero) on the surface. To allow, in general, for such a case, let us specify that *stresses* are prescribed in $6 - p$ dimensions; one of these prescribes the hydrostatic pressure. From the others follows

$$\epsilon_{\lambda}^0 = P_{\lambda\kappa}^{-1} \bar{\sigma}_{\kappa}^0 \quad (5 - p \text{ values of } \lambda \text{ and } \kappa)$$

(4.31)

and

$$d\epsilon_{\lambda} = \epsilon_{\lambda}^0 d\bar{\epsilon}$$

At this point, the entire (average) stress state is known; it can be inserted into $\tau(\sigma)$, if there is such a dependence, and then the *yield surface* is described by the five-dimensional locus of

$$\bar{\sigma}' = \bar{\mathbf{M}}(\epsilon^0, \{\Omega\}) \cdot \bar{\tau}(T, \dot{\Gamma}, \sigma, \epsilon^0) \quad (4.32)$$

while the *plastic potential* is just the locus of $\bar{\mathbf{M}}$. Here, the vector $\bar{\mathbf{M}}$ has been re-introduced for $\sigma^0 \cdot \bar{\mathbf{M}}$ (eqns (4.18) and (4.23)), and its dependence on the *orientation distribution* has been expressed by the set $\{\Omega\}$ of orientations Ω . Furthermore, the dependence of the plastic resistance τ on its primary variables temperature and strain rate, on the stress state, and possibly on the straining direction has been explicitly stated.

4.2.6. Adding Anisotropic Elasticity

So far, we have deliberately ignored elasticity. The strain-increment, whose direction played such a crucial role in determining the stress state necessary for plasticity, was taken to be the *total* (deviatoric) $d\epsilon$. In truth, this should have been the *plastic* strain increment.

The following iterative procedure, in a step-wise calculation, may be used to take account of the change in the direction of the plastic strain-increment $\Delta\epsilon^p$ at a given total strain-increment $\Delta\epsilon^t$, due to the direction of the elastic strain-increment $\Delta\epsilon^e$. Assume first (as we have) that $\Delta\epsilon^p = \Delta\epsilon^t$, and calculate the values of σ' both before and after the step. Then use the difference between these two stress values to calculate the deviatoric part of the elastic strain-increment, and finally subtract it from $\Delta\epsilon^t$ to determine a better approximation of $\Delta\epsilon^p$:

$$\Delta\epsilon^p = \Delta\epsilon^t - \mathbf{S} \Delta\sigma' \quad (4.33)$$

where \mathbf{S} stands for the anisotropic deviatoric elastic compliance (a fourth-rank tensor). All quantities are written in the (stress-free) lattice coordinate system; elastic rotations may safely be neglected.

There is one set of problems where adding elasticity is not an option but a necessity (even though only the plastic response may be of interest): that is when parts of a body, at some time, may not actually be plastic. This could well occur in cases where the single crystal yield surface is very elongated (or even open in some directions of stress space), or is simply very large in some places (e.g. in a two-phase material). None of this would pose any problem if the local stress were completely known: one could check whether it is inside the yield surface or not. But, under prescribed straining conditions, the yield condition could only be verified by assuming the total strain increment to be elastic and checking for the location of the resulting stress. As in all cases, incorporation of rate dependence alleviates the problem also here.

5. EVOLUTION

It is part and parcel of a state-parameter description that the evolution of the parameters must be described: that is how history makes its impact on the future behavior. There have been many discussions of flow-stress evolution and various transients in strain-hardening that make further parameters necessary.⁶⁹ However, there has been very little discussion concerning the effects of the *change* of textures, and thus of average orientation factors, during deformation.⁵⁵ Instead the multiaxial behavior has been described by various postulates. We will compare these with physically expected behavior.

5.1. Texture Evolution

5.1.1. Hardening Rules

The most evident effect of deformation on the calculated yield surfaces of the last section is a change in their *shapes*. This is not accounted for by any of the classical assumptions on 'hardening rules': they have postulated only a change in *size* ('isotropic hardening') and a change in *location* of the yield surface in stress space ('kinematic hardening').⁷⁰

The difference between these postulates and the 'real' behavior, as exemplified in the foregoing section, is not negligible, because of the

developing vertices: while the difference in the absolute value of the resolved stress σ may not vary too much with direction in stress space, the angle between the stress direction and the strain-increment direction varies widely and abruptly. (In the von Mises model, this angle is zero and σ is constant for all directions.)

It would appear that a new formulation that takes some account of *angular yield surface shapes* must be developed. Until such a comprehensive treatment is available, a realistic assessment of yield surface evolution can only be obtained by direct measurement (which is especially difficult near vertices)—or by the measurement or calculation of textures, from which yield surfaces are to be calculated.

To calculate texture changes, one must carry the polycrystal model (Section 4) further: the distortion rate (eqn (4.1)) must be calculated from the slips on all systems, which can be unambiguously obtained by using rate dependence (eqn (3.10)) to invert the yield condition (eqns (4.4) and (4.5)).

In addition to changes in *shape*, there is, of course, a change in *size* of the yield surface: it increases with strain as τ does ('isotropic hardening'; Section 5.2). For two-phase materials, one would, in addition, expect some 'kinematic hardening' (Section 7). Finally, there is some evidence for a phenomenon that may be called 'lateral hardening':⁷¹ a slight additional growth of the yield surface in directions perpendicular to the previously applied straining direction. It would be expected as a consequence of 'latent hardening' in single crystals (see Section 5.2.1).

5.1.2. Texture Hardening and Texture Softening

In eqn (4.15), we defined a scalar stress parameter σ that characterizes the spacing of a tangent plane to the yield surface (and *depends* on the strain-increment direction, i.e. the inclination of the tangent plane). The change of this 'resolved stress' with the scalar equivalent strain increment $d\epsilon$ is an expression of (multiaxial) yield stress evolution. With eqn (4.22) it is

$$\Theta \equiv \frac{d\sigma}{d\epsilon} = \bar{\tau} \frac{d\bar{M}}{d\epsilon} + \bar{M} \frac{d\bar{\tau}}{d\epsilon} \quad (5.1)$$

These are averages over the polycrystal, but we have dropped the bar over the σ . (Also, the possible dependence of $\bar{\tau}$ on strain rate, temperature, etc. has not been explicitly stated in eqn (5.1).) The first term on the right-hand side is due to a change in texture, the second to

a change in dislocation structure. The latter is actually related to the crystallographic shears in the grains, so that eqn (5.1) is better written, with the aid of eqns (4.11) and (4.16), as†

$$\Theta \equiv \frac{d\sigma}{d\varepsilon} = \bar{\tau} \frac{d\bar{M}}{d\varepsilon} + \bar{M}^2 \frac{d\bar{\tau}}{d\bar{\Gamma}} = \bar{\tau} \bar{M}' + \bar{M}^2 \theta \quad (5.2)$$

Here, the standard symbol, θ , has been used for the microscopic strain-hardening rate, and \bar{M}' has been used for the change with strain in the average Taylor factor due to texture development. This change is due not only to the change in orientation of each grain *per se*, but also to the abrupt change in the activated vertex that such an orientation change occasionally entails.

The texture development can cause additional hardening; for example, in the tension (or wire drawing) of fcc metals, a $\langle 111 \rangle$ fiber texture develops, which has the highest Taylor factor for uniaxial tests. On the other hand, it is also possible for a material to exhibit *texture softening* under certain conditions, and that has the most serious consequences: it can lead to instabilities even at a positive (microscopic) strain-hardening rate θ .⁷² The most potent combination for such a development is a vertex in the yield surface with texture softening occurring for one of the strain paths allowed in the vertex.

5.1.3. Path Dependence

Until now, it has been common to treat yield surface evolution as consisting of two aspects: the change in the yield surface (size, location, shape, etc.), and the rate of change with strain of the scalar parameter (or parameters) that describe it. It is, however, realized that this rate of change may itself depend on the strain path. For example, the texture that develops in torsion is different from that in tension, and the latter is different from that developed in compression. The change in yield surface shape will consequently also be different: sometimes the direction of prestraining is the hardest, sometimes not. (This is possible even when *microscopic* hardening is isotropic.) Unfortunately, there seems, at present, no general way of accounting for all these different possibilities—except by explicit texture calculation (or measurement) at every step.

† Note that this hardening is scalar in nature; an introduction of tensor stress-rates is not necessary when the lattice orientation change has been accounted for in this way (and when there is no kinematic hardening). However, Θ may well depend on ε^0 (or σ^0).

A path dependence of macroscopic strain hardening can, in addition, be due to a path dependence of microscopic hardening. For example, the rate of dislocation accumulation appears to be smaller in torsion than in tension, perhaps because of the smaller number of slip systems that are significantly active.⁵⁵

Both of these types of path dependence can, finally, make themselves felt during *path changes*: here one would expect significant *transients*. For example, since the range of orientation space that is filled after some heavy deformation is quite small, the re-orientation under the new test conditions may not lead to the same end texture as would be expected from a random start—or it may take much strain to convert to the new ‘equilibrium’ texture.

More potently, dislocation structures accumulated under one set of conditions tend to be unstable under other conditions; thus, one would expect a temporary softening, possibly an instability, upon a change in straining path.⁷³

This is another set of problems that needs major theoretical and experimental attention.

5.2. Substructure Evolution

5.2.1. Strain Hardening and Latent Hardening

The microscopic (and major) aspect of strain hardening is the change of the local flow stress with strain. This is what is primarily meant by the term strain hardening:

$$\theta \equiv \left. \frac{\partial \tau}{\partial \Gamma} \right|_{\dot{\Gamma}, T} = \theta(\dot{\Gamma}, T, \tau, \epsilon^0) \quad (5.3)$$

Note that it is written in differential form, and as a function of state parameters and current conditions only.† Any relation that involves strain as an explicit, integral variable (as most of the common ones) is subject to additional path dependences.^{9,75-77}

Equation (5.3) has been written in terms of a single grain. Truly, the average behavior of many grains (not considering orientation effects) should be described;⁷⁸ we have found⁵⁵ that the different ways of averaging do not change the result significantly, presumably because

† θ could also depend on the current state of stress; this may, for example, explain its orientation dependence in single crystals.⁷⁴

the spread of flow stresses in different grains at any one time does not cover a wide range of θ 's.

The strain hardening rate, eqn (5.3), has been written in terms of Γ , the algebraic sum of shears (or, more generally, the work conjugate of τ). This is the quantity that can be directly related to the macroscopic strain by the Taylor factor (eqn (4.11)). Equation (5.3) does not imply that θ is unique for all combinations of slip increments that make up the same $d\Gamma$ (as was originally assumed by Taylor⁴⁶); in fact, the inclusion of the straining direction ϵ^0 in the list of arguments explicitly allows for such non-uniqueness. Unfortunately, not much is known about the general behavior under the type of constrained flow (multiple slip) that occurs in grains of a polycrystal.

There have been attempts to describe single-crystal hardening in more detail by allowing for different hardening coefficients for different slip system interactions. Specifically, if two slip systems are labelled s and t , the interaction has been written in matrix form:

$$d\tau^s = h^{st} d\gamma^t \quad (5.4)$$

(and sometimes the orientation-change part of hardening, from eqn (5.2), has been built into this form^{79,80}). This is probably grossly oversimplified. For a realistic description, one must at least separate two effects:⁸⁰ the ratio of the flow stresses on different systems *at the same structure* (this is the 'latent hardening ratio'); and the rate of development of various components of dislocation densities with slip on one slip system.

Equation (5.4) would have serious (beneficial) consequences on the theory of polycrystal plasticity if the hardening matrix were positive definite,⁸¹ that is, roughly speaking, if the off-diagonal terms were smaller than the diagonal terms. The part of h^{st} that is due to the latent-hardening ratio can be tested in single crystals, operating single slip systems alternately.^{82,83} The observations are that latent hardening is always equal to or *larger* than direct hardening. (It is never much greater: typical latent-hardening ratios are at most 1.4, except after easy glide in single slip.)

In summary, the global formulation (5.3) is, for now, as realistic as anything available.

For a quantitative description of strain hardening of any particular material, it is best to plot the differential relation (5.3). In the spirit of our description of polycrystal behavior, this should actually be a single-crystal relation. However, representative grains in a polycrystal

are not likely to behave exactly the same as free single crystals. The most important difference is that they are forced to slip on many slip systems, whereas most free single crystals deform in single slip. In fcc metals, the most appropriate single-crystal curve to use is that for tension or compression in a $\langle 111 \rangle$ direction,⁸⁴ since here slip actually occurs on many slip systems throughout the straining.

Figure 13(a) shows a set of $\theta(\tau)$ diagrams for Al $\langle 111 \rangle$ crystals at different temperatures,⁸⁵ and Fig. 13(b) demonstrates the behavior of polycrystalline pure Ni and two Ni alloys in torsion.⁸ The latter is, in our terminology, a $\Theta(\sigma)$ diagram: textural effects have not been taken out. From a stress-dependence point of view, Figs 13(a) and 13(b) exhibit similar behavior: plasticity begins at $\theta \approx \mu/200$ ('stage II'); then θ decreases in a gradual fashion ('stage III'). In polycrystals at very large strains, one generally finds an eventual stage of constant, very low hardening rate, of the order of $\mu/5000$ (or at $\Theta \approx \sigma/4$ for two-phase materials).⁵⁻⁸ This region (called 'stage IV'⁶) is little understood,⁷⁸ and we exempt it from discussion here.

The decrease of θ with τ may sometimes be approximated, over a meaningful range, as being linear; then, this corresponds to a Voce law of the stress/strain curve.^{86,13} More generally, however, it is the actual curve $\theta(\tau)$ that must be tabulated if the behavior of a particular material is to be described accurately; no generally applicable function is likely to be found.

5.2.2. Dynamic Recovery and Saturation

An interesting result of crystal plasticity investigations is that the strain-hardening rate consists of two components^{12,13,76}

$$\theta = \theta_a - \theta_r(\dot{\Gamma}, T, \tau, \epsilon^0) \quad (5.5)$$

The first is an *athermal hardening* component, the second *dynamic recovery*; all the temperature and rate sensitivity lies in the latter (and some dependence on the straining direction is also admitted).

Athermal hardening was first found in free single crystals in single slip and was there labelled 'stage II'.⁸⁷ Its prime characteristics are, however, present in multi-slip deformation and in polycrystals;¹² they are: a value of about $\mu/200$ (within, say, $\pm 50\%$), and an insensitivity to temperature, strain rate, and material (within the same class, such as cubic, single-phase materials). When this stage is extensive, θ_a is also virtually constant with stress; more generally, it is defined as *limiting behavior at low strains and low temperatures*. The value $\mu/200$

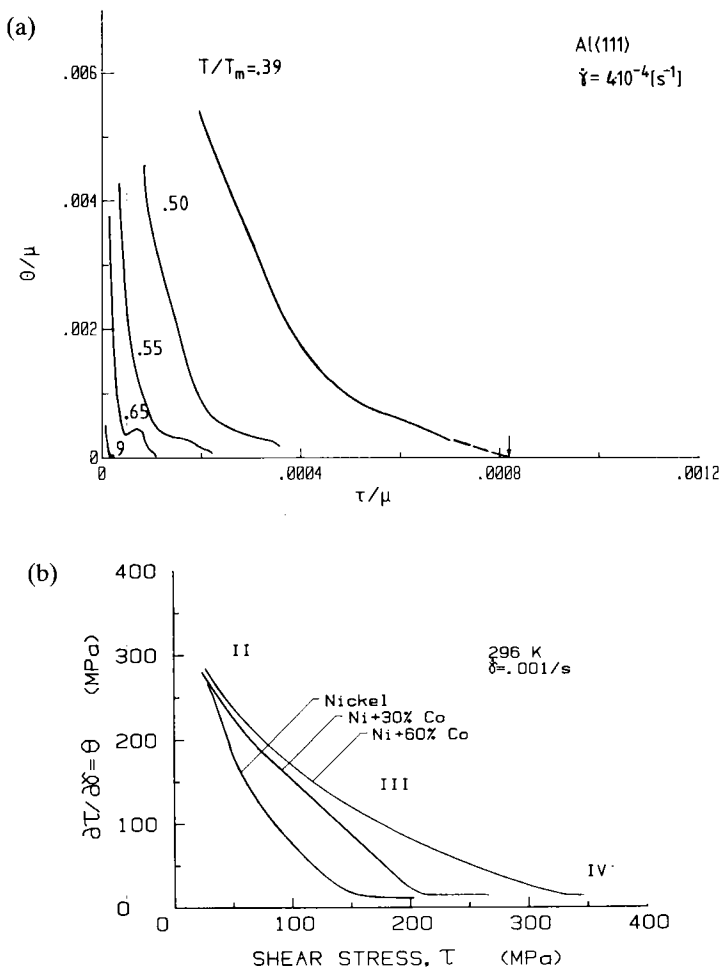


Fig. 13. The strain-hardening rate θ versus the flow stress τ , both normalized by the shear modulus μ and plotted as an implicit function of strain: (a) aluminum $\langle 111 \rangle$ single crystals in tension for various temperatures;⁸⁵ (b) nickel and nickel-cobalt polycrystals in short torsion.⁸ There is a general decrease that may, over some regime, be approximated as linear and extrapolated to some 'saturation stress' limit; however, the strain-hardening rate eventually reaches a low constant value in many cases.

is what led to the statement, in Section 2.1.1, that $d\sigma/d\epsilon \leq E/50$: the square of the Taylor factor is of order 10, and $E/\mu \approx 2.5$.

The athermal hardening component has been explained on the basis of a geometrical-statistical dislocation storage rate.⁸⁸ It leads to a more-or-less random (tangled) arrangement of dislocations. *Dynamic recovery*, on the other hand, is associated with the rearrangement of the previously stored dislocations, under the action of stress, temperature, and the effects of continued straining. It leads to cellular, and eventually subgranular, substructures.^{5,89}

The curves in Fig. 13(a) exhibit an example of an interesting scaling relation: to a good approximation, all the temperature dependence is in a stress-scale factor. This type of behavior is found quite generally, also for effects of strain-rate and even solute concentration (Fig. 13(b)). Thus, one may write (with a yield stress τ_0)

$$\begin{aligned}\theta &= \theta_a - \theta_r \left(\frac{\tau - \tau_0}{\tau_s} \right) \\ \tau_s &= \tau_s(T, \dot{\Gamma})\end{aligned}\quad (5.6)$$

When the function θ_r is linear, we recover the Voce law, and τ_s may be a saturation stress; in general, it is just a stress-scaling factor.

The rate- and temperature-sensitivity of τ_s is different, both in cause and in magnitude, from the rate- and temperature-sensitivity of the flow stress (compared at constant structure). It is, in fact, about an order of magnitude larger. This is particularly important in the *limit of steady-state or flow-stress saturation*: if the limit $\theta = 0$ can be operationally defined by extrapolation (even if it may not actually be reached), then the rate- and temperature-dependence of this steady-state limit is controlled by that of dynamic recovery, not by that of glide kinetics.

The stress exponent (eqn (2.2)) can now be expressed as

$$n = \left. \frac{\partial \ln \dot{\epsilon}}{\partial \ln \sigma} \right|_{T, \Theta=0} = \left. \frac{\partial \ln \dot{\Gamma}}{\partial \ln \tau} \right|_{T, \theta=0} \quad (5.7)$$

A useful approximation to this steady-state value is that at the 'ultimate tensile strength', where $\Theta = \sigma$ (not $\Theta = 0$). Note that this relation between stress and strain-rate is due to thermal activation; thus, n should be expected to be temperature dependent (see Section 3.2.4). Within our 'interest space', it is of order 10, varying from about

20 or 30 at room temperature (depending on material) to a very-high temperature limit of 3 or 4.⁹⁰

5.2.3. Temperature/Material Scaling

The temperature dependence of dynamic recovery, and thus of strain-hardening, can be expressed as that of the stress-scaling parameter τ_s (eqns (5.6)). An example of such a description is given in Fig. 14, which refers to a series of pure fcc $\langle 111 \rangle$ single crystals; the saturation stress from a 'long' extrapolation was used for τ_s ,⁸⁵ and it is plotted as τ_s/μ versus $kT/\mu b^3$.

The stress and temperature were normalized by a further, constant scaling factor each, depending only on the material. It was found that then all these four materials could be brought into coincidence. The two scaling factors s^* and t^* (normalized to Al) in fact were correlated with each other and with the parameter $\chi/\mu b$, where χ is the

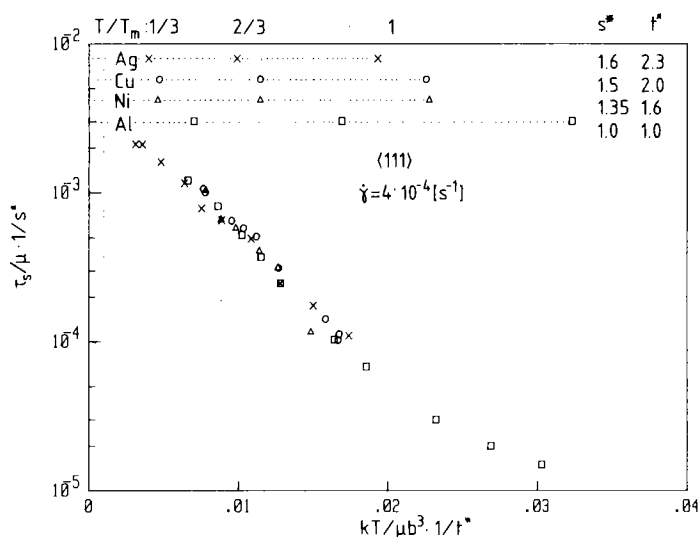


Fig. 14. The saturation stress τ_s (or any other scaling parameter for the stress axis in diagrams like Fig. 13) decreases approximately linearly with temperature, in a semi-logarithmic plot. Different materials may be unified by using scaling parameters s^* and t^* for the axes on this plot, which correlate with each other and with the stacking-fault energy.⁸⁵ Note also that there is a maximum saturation shear stress of about $5 \times 10^{-3} \mu$ at zero temperature for these pure fcc materials.

stacking-fault energy (SFE). This is a quantitative expression of the well known fact⁹¹ that dynamic recovery is faster for high-SFE materials than for low-SFE materials, at the same homologous temperature. In fact, some parameter linked to $\chi/\mu b$, not to T_m , provides the appropriate normalization of temperature for plastic flow.

A surprising observation from Fig. 14 was⁸⁵ that this same normalization appears to hold over the *entire* temperature range, without ever requiring the introduction of parameters related to self-diffusion. In fact, Fig. 14 indicates that the functional dependence is essentially straight:

$$\ln \frac{\tau_s}{\mu} = \ln \frac{\tau_{s_0}}{\mu_0} + \frac{kT}{\mu b^3 g_s} \ln \frac{\dot{\gamma}}{\dot{\gamma}_{s_0}} \quad (5.8)$$

where the two scaling parameters are labelled τ_{s_0} and g_s for each material. (They are proportional to s^* and t^* .) The expected strain-rate dependence is also introduced. The strain-rate constant $\dot{\gamma}_{s_0}$ is not well established and probably depends on τ_s .⁸⁵ It was measured to be of order 10^5 s^{-1} in Al,^{13,85} and of order 10^{10} s^{-1} in Cu.⁹²

The first interesting observation to be made about eqn (5.8) is that there exists an absolute maximum saturation stress, corresponding to mechanical collapse of the dislocation structure (in the zero temperature limit). Its value is approximately $10^{-2} \mu_0$ (as a crystallographic shear stress), for all fcc metals.¹³

Equation (5.8) expresses that the activation energy depends logarithmically on the stress, and a power-law stress/strain-rate relation is obeyed, with the exponent (eqn (5.7)) independent of stress and strain-rate, but $n \propto 1/T$.

A detailed scrutiny of this abstraction⁸⁵ revealed that, above $T/T_m \approx 2/3$, the data can also be described in a diffusion-related way. For the case of aluminum, it was in fact shown from strain-rate data over a sufficient range that the relationship

$$n = n_0 + \frac{\mu b^3}{kT} g_s \quad (5.9)$$

with $n_0 \approx 3.7$ provides an even better fit over the entire temperature range, and gives a constant high-temperature limit of the activation energy, which agrees with that of self-diffusion.⁸⁵ In other materials, similar superposition laws for the steady-state stress-exponent have also been found;⁹³ the data tend to fit a value $n \approx 4$ (although 3 would be easier to explain^{1,94}).

In summary, eqn (5.8) gives a good first-order description of dynamic-recovery kinetics, at least up to $T/T_m = 2/3$. For a more accurate tabulation, especially for high-temperature applications, one must record the function S in

$$\ln \frac{\tau_s}{\mu} = \ln \frac{\tau_{s0}}{\mu_0} + S \left(\frac{kT}{\mu b^3} \ln \frac{\dot{\gamma}_{s0}}{\dot{\Gamma}} \right) \quad (5.8')$$

with $\dot{\gamma}_{s0}$ either constant or possibly dependent on τ_s/μ . In any case, it seems appropriate from these considerations to leave the value and functional form of the activation energy open, and not prejudice the data by normalizing with the self-diffusion energy Q ; the Fisher⁴² parameter $kT \ln(\dot{\epsilon}_0/\dot{\epsilon})$ should be preferred over the Zener parameter $\dot{\epsilon} \exp(Q/kT)$, because the result is much less sensitive to the value of the free parameter ($\dot{\epsilon}_0$ vs. Q) chosen for the evaluation.

5.2.4. Creep

This term is used in two ways. Its definition, for us, is deformation at constant stress (or at least constant load); then the observed behavior is strain versus time, or strain-rate versus strain—different from a 'stress/strain test'. But the term 'creep' is also used for 'slow deformation at high temperature'—because these are the conditions under which its observation is easy.

In the first ('true') definition, 'creep' and 'plasticity' are merely different expressions of the *same* material behavior. That is why this book is one of the efforts to attain a 'unified' description of creep and plasticity. We, in fact, use the term plasticity for both: plasticity is, *ab initio*, allowed to be rate-dependent.

But there is another difference: creep is usually studied *near steady state*, with the initial loading strain and the transient creep strain held to about the same order as the 'steady' strain or less. Thus, the behavior is almost always dominated by recovery (dynamic or static); this may bring in special problems related to 'hardening transients', which we will discuss in Section 5.2.6 but ignore for now. In addition, there is a good possibility that the 'rearrangement strain' that occurs during dynamic recovery^{95,96} contributes substantially to the total strain-rate near steady state.^{2,12} Only tentative descriptions have been given for its relation to flow stress and strain-hardening.

The material response measured in a creep test is the *creep deceleration*

$$\delta = - \left. \frac{\partial \ln \dot{\epsilon}}{\partial \epsilon} \right|_{\sigma, T} \quad (5.10)$$

We have written it, in the usual form, in terms of the macroscopic variables σ and $d\epsilon$, not in the microscopic τ and $d\Gamma$; this is acceptable because the strains during typical creep deformation in metals are not large enough to cause any significant *change* in texture; thus, there is only a constant orientation factor to be taken care of. Note, however, that the symbols σ and $d\epsilon$ are in fact supposed to mean the *equivalent* stress and strain-rate defined in Section 5.2.1.

An unfortunate fact is that in creep the material parameter (δ in eqn (5.10)) can only be obtained by *double* differentiation of the experimentally obtained $\epsilon(t)$ data; for this reason, stress/strain tests are easier to evaluate: only a single differentiation is needed to obtain θ or m .

5.2.5. A Differential Constitutive Equation

Until now, we have discussed the material behavior in terms of two separable aspects: its properties in a given state (having a given structure); and the evolution of this state/structure with strain. The first is typically described by a relation $\tau = \tau(\dot{\gamma}, T, \hat{\tau})$, such as eqn (3.8), the second as one or more *differential* relations such as eqn (5.3).

One could, of course, differentiate the first relation and insert it into the second. The results could, for example, be

$$d \ln \tau = \left. \frac{\partial \ln \tau}{\partial \ln \dot{\gamma}} \right|_{\hat{\tau}} d \ln \dot{\gamma} + \left. \frac{\partial \ln \tau}{\partial \ln \hat{\tau}} \right|_{\dot{\gamma}} \frac{d \ln \hat{\tau}}{d \gamma} d \gamma \quad (5.11)$$

where we have left out the temperature as an explicit variable. This describes the dependence of the flow stress on both strain rate and strain in differential form.

The advantage of such a formulation comes when one writes it in a phenomenological form, for example

$$d \ln \sigma = m d \ln \dot{\epsilon} + H d \epsilon \quad (5.12)$$

Here, we have shifted to the macroscopic variables. The coefficients are measurable; in particular†

$$H \equiv \left. \frac{\partial \ln \sigma}{\partial \epsilon} \right|_{\dot{\epsilon}} = \frac{\Theta}{\sigma} \quad (5.13)$$

† H may be called the *work-hardening* rate, since it equals $d\sigma/dW$, as distinct from the *strain-hardening* rate $\Theta = d\sigma/d\epsilon$. Which of these is preferred depends on the application. Note that both the strain and the work are used in differential form only.

If eqn (5.12) is viewed as the *primary* statement of constitutive behavior, and not as a derivative of the state formulation, as in eqn (5.11), then *it is not necessarily integrable*. One can, however, state the precise conditions under which it is:⁴⁴ when m and H are functions of σ and $\dot{\epsilon}$ only, not of any further, 'hidden' variables; then an integrating factor for $d\epsilon$ can be found, and the integral of the combination is a state parameter—there is only one.

There are many reasons why the differential constitutive equation (5.12) might not be integrable, why there should be 'hidden parameters'. For example, we know that, in general, the texture will change with strain; then H will depend on path, even when both σ and $\dot{\epsilon}$ are fixed. Similarly, σ could depend on grain size—say, in particular, on the minimum dimension of the grains; this changes with strain, in a path dependent manner. In both of these cases, one could add extra terms to eqn (5.12). Unfortunately, no macroscopic way is available (at least as yet) to separate changes in these variables. On the other hand, fortunately, they are likely to become important only at very large strains, where regular strain hardening has virtually saturated out. Thus, one may be able to deal with a single state parameter at a time, but change their identity at some definable demarcation line.

Finally, there is evidence for another state parameter associated with dynamic recovery, and we will treat this in detail in the next section. It may be important for creep. If it is not, eqn (5.12) already describes creep in the 'primary' regime (and in the steady-state limit): comparison of eqns (5.10) and (5.12) shows that

$$\delta = H/m \quad (5.14)$$

The *instantaneous* (iso-structural) rate sensitivity m enters to translate stress/strain data into strain-rate/strain data.[†] Again, when $\delta = \delta(\sigma, \dot{\epsilon})$ only, a single state parameter and an 'equation of state' exist; otherwise, δ must be tabulated for different paths. But it is always measurable, by the operational definition of eqn (5.10).

In summary, the differential constitutive equation (5.12) is 'unified' in that it describes both creep and plasticity; and it is general in that it does not presume the existence of a *single* state parameter or an

[†] The best operational definition of m is in fact H/δ , since H and δ are well defined as rates of change at any instant, whereas m describes the result of an abrupt change 'at constant state', which requires back-extrapolation for the measurement.³

(integral) 'equation of state'. Since it is a (first-order) differential equation, it requires an initial value pair; this is equivalent to a parameter of the current state.

5.2.6. *Hardening Transients*

We already sketched in Fig. 3 (b and c) that fairly long transients sometimes occur after a strain-rate or temperature change; this is in fact a characteristic of dynamic recovery. Qualitatively, it has been explained as being due to an exhaustion, under the current conditions, of recoverable dislocation arrangements: when conditions 'improve' (lower rate, higher temperature) some extra recovery occurs; in the reverse case, some athermal dislocation storage must first take place before recoverable tangles are formed again. Thus, it is really a transient in the hardening rate (not in glide kinetics).¹⁰

The transient is evidence for some structural change that corresponds to the evolution of a 'second' state parameter (other than the flow stress, and not counting texture, grain size, and others). This state parameter seems to attain a steady-state value in equilibrium with the external conditions and the other state parameters after about 3–5% strain. It is now a question of judgment, for a specific application, whether one wants to describe this short-term evolution or not. If not, back-extrapolation from larger strains would give a self-consistent description—and a value of $\Delta\sigma$ for the prescribed $\Delta \ln \dot{\epsilon}$ or ΔT that reflects the new *evolution* rate (and is much larger than the 'instantaneous' value, due to glide kinetics).

However, this hardening transient may be quite important in the neighborhood of steady-state deformation—and therefore for creep. It is not clear, at this point, whether transient creep (of the 'normal' kind) is more directly connected to the main part of strain hardening or more to the exhaustion of dynamic recovery—probably both, under different limiting circumstances.

Since we interpret this transient as due to the evolution of a *hardening* parameter, it seems appropriate to write a differential equation for H (which, in a way, is a *second-order* differential equation in the flow stress, and thus leads to *two* state parameters, even if it is path independent):

$$dH = -C d \ln \sigma + B d \ln \dot{\epsilon} - E d\epsilon \quad (5.15)$$

The coefficients have been defined so that they are normally positive. Without the evolution term at the end, the equation has been used

before to describe the dependence of work hardening on stress and strain-rate.^{97,98} But without this evolution term, the hardening slopes at points A and B in Fig. 15 should be the same; the whole point of eqn (5.15) is to describe the fact that this is not true during some transient regime.

The new coefficient E can be defined as

$$E \equiv \left. \frac{\partial H}{\partial \epsilon} \right|_{\sigma, \dot{\epsilon}} \quad (5.16)$$

Combining eqn (5.16) with eqn (5.12), we have

$$dH = -(E + CH) d\epsilon + (B - Cm) d \ln \dot{\epsilon} \quad (5.17)$$

The coefficient of $d\epsilon$ is the *curvature* of the $\ln \sigma$ vs. ϵ diagram (at constant $\dot{\epsilon}$), and E is the excess of this curvature in the transient over that during 'steady-state work hardening' (i.e. steady state of the *second* state parameter). The second term on the right-hand side is the rate sensitivity of work hardening; the contribution Cm comes from glide kinetics and is negligible with respect to that due to evolution kinetics, B .

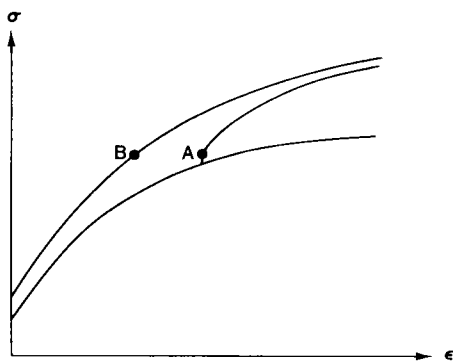


Fig. 15. Two continuous stress/strain curves and one with a change, near A, from one set of conditions (such as strain rate, temperature) to the other. If there were but a single state parameter, the 'future' starting from point A would look the same as from point B. The fact that this is generally not true is interpreted as a transient and as evidence of the need for an additional state parameter if this transient is to be described.

A completely equivalent 'second-order' evolution equation can be written in terms of the creep deceleration:³

$$d\delta = -c d \ln \sigma + b d \ln \dot{\epsilon} - e d\epsilon \quad (5.18)$$

which leads, with eqns (5.9) and (5.11), to

$$d\delta = (e + b\delta) d\epsilon + (b/m - c) d \ln \sigma \quad (5.19)$$

Here, $c \ll b/m$.

In (complete) steady state, the terms in $d\epsilon$ in both equations (5.15) and (5.18) vanish, and we have for the stress exponent:

$$n = \left. \frac{\partial \ln \dot{\epsilon}}{\partial \ln \sigma} \right|_{H=0} = \left. \frac{\partial \ln \dot{\epsilon}}{\partial \ln \sigma} \right|_{\delta=0} = \frac{C}{B} = \frac{c}{b} \quad (5.20)$$

In steady state itself, there is no state parameter: they are all in equilibrium with external variables. *Near* steady state, one should expect *one* state parameter to be sufficient. It is an important question for a unified constitutive description at the present time whether this state parameter is the flow stress (so that eqn (5.12) can be used) or the hardening rate (which might be describable by eqn (5.15)). In the first case, it is the dislocation *density* that adjusts, in the second the dislocation *arrangement*.

5.2.7. Conclusion

The evolution of the state is best described by a strain-hardening rate Θ as a function of state variables and parameters. Texture changes during straining can significantly contribute to Θ , especially at intermediate strains where the changes are significant and have not yet saturated. No simple description of this influence is as yet available, nor is one likely to be found. Calculations in terms of polycrystal plasticity simulation are now possible (if the initial texture is known) and they may be necessary when anisotropic effects are of prime importance.

The 'physical' (rather than textural) hardening rate in the grains may usually be approximated by that of a representative ('average') grain. Its dependence on stress is primarily due to dynamic recovery, and it can be scaled by a stress parameter (the extrapolated 'saturation stress', or some other), which itself depends on temperature and strain-rate. For all of these dependences, simple linear relations in certain plots are often a good first approximation—but they are never

based on a compelling physical mechanism. In any case, the combination of variables that should be plotted or tabulated against each other is given with some generality.

An alternative that may sometimes be preferable to a description in terms of state parameters is an entirely differential constitutive relation in terms of external variables only (but not necessarily integrable). The number and order of such differential equations must be sufficient for the number of state parameters necessary, i.e. the number of transients identified as important for the material and conditions to be treated.

6. INTERNAL STRESSES

The term 'internal stress' has been used in many different meanings in plasticity.⁹⁹ These fall into two major classes: first, a 'true' stress, tensorial in character and with sign, to be added to (or subtracted from) the *applied stress*, and measurable in principle through the difference of the local elastic distortions from those expected on the basis of the applied stress alone. The spatial average of every component of this stress must be zero, but it may nevertheless be relevant, especially when the positive and negative contributions are spatially ordered. This is equivalent to the 'residual stress' used on a specimen scale to describe the consequence of macroscopically non-uniform deformation; but it also occurs as a consequence of microscopically non-uniform deformation, as it occurs in all polycrystals and especially in 'mixtures' of grains of different materials. It also occurs in 'two-phase materials', in which different components occur in each grain. This can be a strong effect and will be discussed first.

The other class of 'internal stresses' or 'back-stresses' that has been used frequently is an athermal and rate-independent contribution to the (scalar) *flow stress* or glide resistance; it need not be additive (though this is implied by using the term 'internal stress').

Finally, the term 'internal stress' has occasionally been used to describe local stress redistributions due to internal rearrangements caused by an applied stress, such as by the bowing-out of dislocations or dislocation walls.¹⁰⁰ As was discussed in Section 2.1.1, we wish to treat as state parameters only quantities that can be assessed in the unloaded state: they cause all subsequent behavior.^{10,89}

6.1. Two-phase Materials

6.1.1. The Orowan Stress

Imagine a single-crystal matrix with a dispersion of spheres of a second phase that causes no misfit (and thus no 'true' internal stresses) and has the same elastic properties as the matrix; the only difference between the two phases is in their yield strengths. Then, at an applied stress between the two yield strengths, there will be a contribution to the yield strength of the aggregate due to the need for repeated, transient bowing-out of the dislocations to surround the particles. If the spacing between the particles is L , this 'Orowan stress'¹⁰¹ is approximately¹⁰²

$$\tau_o \approx \mu b / L \quad (6.1)$$

It is quite insensitive to thermal activation (it is 'athermal', rate-independent), and it may superpose with other contributions to the flow stress either additively (e.g. with solution hardening¹⁰³) or quadratically (with strain hardening^{2,4,104}).

To summarize: the Orowan stress is an example of a scalar, athermal contribution to the yield strength (additive or not), but it does not correspond to any internal stress measurable in the crystal before loading.

6.1.2. Strain Hardening

The remnants of the dislocations that bowed around the impenetrable particles in the above example accumulate around the particles and make for an extra contribution to strain hardening. These extra dislocations have been described as 'geometrically necessary':¹⁰⁴ they accommodate, in the matrix, the (true, tensorial) internal stresses caused by a now misfitting particle. In fact, their density is initially proportional to the elastic strain in the particle, and thus to the plastic strain in the matrix. Their effect on strain hardening is essentially through this (scalar) dislocation density ρ ; since $\tau \propto \sqrt{\rho}$, the strain hardening they cause (if it is alone) is truly parabolic: $\tau \propto \sqrt{\gamma}$. This is, again, a (scalar) contribution to the flow stress (as rate insensitive as all strain-hardening contributions).

With increasing strain, the statistical accumulation of dislocations (both because of the above 'geometrically necessary' ones and because of all other obstacles) usually dominates strain-hardening behavior; thus, parabolic hardening is typically observed only until θ has decreased to the value θ_a (Section 5.2.2).

6.1.3. Forward and Backward Internal Stresses

When particles have to deform elastically while the matrix deforms plastically, a 'forward' internal stress must be generated inside each of them: forward in the sense that it is in the same direction as the macroscopic plastic strain. Note that it is the strain that counts, in this case, not the stress; for example, in a single crystal undergoing primarily single slip, the internal stress in any imbedded, non-deformable particles is of the same character as the shear strain caused by slip (even if the stress might be uniaxial tension), and in the same direction (assuming elastic isotropy).

Since all true internal stresses must average out to zero through the aggregate, it follows that the average internal stress in the matrix must be 'backward': against the prevailing strain direction. It is, for some applications, helpful to consider this *average* back-stress in the matrix; it has also been called 'image stress'.¹⁰⁵ It is a true, tensorial, internal stress. Its magnitude is proportional to the volume fraction of particles; typically, it is about 10–20% of the applied stress, and this part should truly be subtracted from the applied stress, *tensorially*, to get the glide resistance.

This is called the *kinematic hardening* component in the phenomenological plasticity literature and often labelled α , signed as a back-stress; then, the yield condition becomes, instead of (4.32):

$$\sigma' - \alpha = \bar{M}\bar{\tau} \quad (6.2)$$

where $\bar{\tau}$ is now the plastic resistance due to both particles and dislocations stored (and may include a contribution from τ_0).

6.1.4. The Bauschinger Effect

Since the plastic deformation in the matrix does not reverse upon unloading, the internal-stress distribution discussed above will essentially remain. Thus, it will also influence reverse loading. Of course, it must eventually change sign; but a reverse plastic strain of the same order as the previous forward strain is needed to achieve this loss of memory. Thus, there should be, and is, a significant Bauschinger effect in two-phase materials.¹⁰⁶ It seems that no systematic investigations of its tensor character have been undertaken as yet.

Figure 16 shows stress reversals in an Al–Cu alloy heat-treated in two different ways:¹⁰⁷ one to keep the alloy single-phase (solution treated), one to make it two-phase (θ'). The qualitative difference between the two 'Bauschinger effects' is obvious.

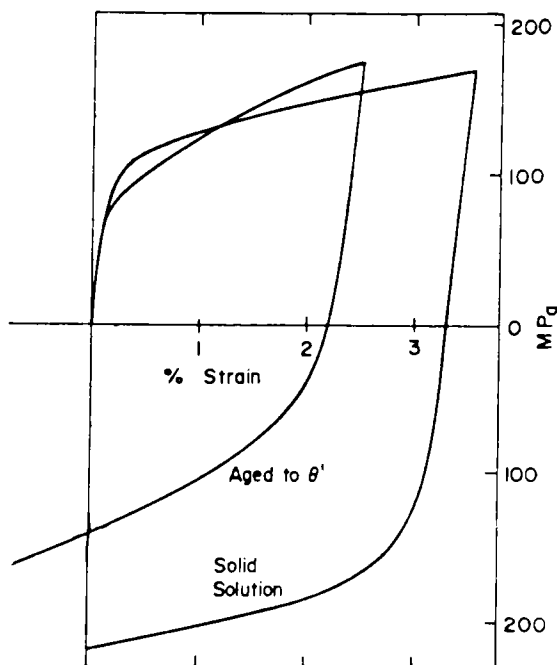


Fig. 16. Stress reversal behavior of single-phase (solution hardened) and two-phase (θ') Al-Cu alloy.¹⁰⁷ Only the latter is a 'permanent' Bauschinger effect, the former is a strain-hardening transient.

'Kinematic hardening' was originally invented to describe Bauschinger effects.⁷⁰ In the spirit we use here (Section 2.1.1), it should best be used *only* for the type of 'permanent' softening upon reverse loading as is observed in two-phase materials: it is a hallmark of a true internal-stress contribution to the flow stress. The substantial lowering of the proportional limit (sometimes involving a change of sign) is better described as a strain-hardening transient (Section 2.1.3).

6.1.5. Recovery

Both the internal-stress build-up and the unusually rapid accumulation of dislocations cause a high rate of energy storage in two-phase materials. This expresses itself in unusually effective recovery: upon annealing and reloading under the same conditions, there is an initial flow stress decrease of the order of 50% in two-phase materials (as

compared to $\sim 10\text{--}20\%$ in single-phase materials), and again a much longer transient to standard behavior. This is another characteristic of materials with 'kinematic hardening', and can be used as a diagnostic test.

6.1.6. Conclusion

There is a class of materials, typified by dispersion-strengthened single crystals, in which internal stresses of opposite sign develop, with deformation, in the two phases. Their existence can be identified by a strong, 'permanent' softening upon either reverse loading or recovery. In addition, these materials show an athermal contribution to the glide resistance at yield, as well as initial parabolic strain-hardening. Thus, they should be described by both a 'kinematic' hardening component (to be subtracted tensorially from the applied stress) and a scalar one (to be accounted for in the plastic resistance). We believe this to be the *only* class of materials where a kinematic hardening component is justified; but note that even here the scalar resistance dominates quantitatively.

6.2. Single-phase Materials

6.2.1. Polycrystal Effects

The interactions between grains in a polycrystal cause internal stresses that vary roughly with a wavelength of the grain size. While these are true internal stresses, tensorial in nature, they are already made use of in all modern theories of polycrystal plasticity: namely such as to activate enough slip systems to satisfy compatibility. They are the reason for setting the effective orientation factor of polycrystals to something larger than the average of that of free single crystals. Thus, they should not be counted again as an internal stress of the kinematic-hardening kind. The same is true for the effects of the so-called geometrically necessary dislocations in polycrystals,¹⁰⁸ which are just another way of describing these very same stresses.⁷⁸

In addition to these 'true' but irrelevant internal stresses, there is generally assumed to be some contribution to strain hardening that specifically relates to polycrystals, presumably due to an extra dislocation accumulation near grain boundaries, more 'statistical' than 'geometrical' in nature. This is probably the main cause of the mild grain-size effect in pure fcc materials.⁷⁸ Its rate sensitivity should be similar to that of other strain-hardening effects. Indeed, the rate-

sensitivity behavior of polycrystals and single crystals is quite similar, at least in pure fcc metals.¹²

Finally, there is a 'back-stress' effect that leads to the 'Hall-Petch' grain-size dependence of the polycrystal yield stress. It is strong only in materials in which dislocation generation or unlocking makes an important contribution to the flow stress (and is often associated with a sharp yield drop). The tensor character of this stress has not been investigated; probably, it is a scalar contribution to the glide resistance. It may sometimes have 'athermal' character, compared to more activatable contributions, but can also be very rate sensitive. In any case, it does not qualify as a kinematic hardening component.

6.2.2. *Forward and Backward Internal Stresses*

There are, of course, true internal stresses in *any* material; e.g. every dislocation is a source of a (tensorial) stress field, each component of which averages out to zero over the body (requiring certain 'image stresses' in finite bodies). In most materials—those that form dislocation tangles and cell walls of the 'thick', disordered kind—these internal stresses are actually ordered: they tend to be 'forward' inside the tangles or cell walls, 'backward' in the more-or-less dislocation-free cell interiors.⁹⁵ This distribution has some similarities to that in two-phase materials. However, its effects on macroscopic properties are quite different.

The polarity of the internal stresses in cellular dislocation structures has been made use of in a number of ways.^{89,109} First, the forward stresses inside cell walls provide the driving force for dislocation rearrangements during dynamic and static recovery.^{95,110,111} Second, the backward stresses in the cell interiors must be important during loading and unloading: during small-strain, contained plastic flow.^{96,112} Finally, the very existence of the polarity makes some kind of asymmetry between forward and reverse straining possible.⁸⁸

Where these internal stresses are different from those in two-phase materials is in their effect on the macroscopic flow stress: we claim they have no effect at all in single-phase materials. On a mechanistic basis, this is plausible because of the areal nature of glide in these materials:⁸⁹ the 'critical gates' through which dislocations must pass to achieve long-range slip^{9,88} are precisely at the saddle points between the regions of forward and backward internal stresses, and are thus at least approximately at zero internal stress.

More potent, perhaps, is a macroscopic argument: the very fact that

the macroscopic flow stress *after any transient* is unaffected, in single-phase as opposed to two-phase materials, by a reversal of straining or by recovery is in itself evidence that internal stresses are not relevant for this particular property—and is, in fact, another justification for using the *macroscopic flow stress* as an important parameter to describe the plastic state.

A powerful tool to ascertain the presence of any separable, athermal contribution to the flow stress (whether tensorial or scalar) is the Haasen plot introduced in Section 3.1.7: when the rate sensitivity $m\dot{\epsilon}$ of the flow stress is proportional to the flow stress itself (i.e. when the 'Cottrell-Stokes law' is obeyed), then there is no such separable component—and this is the case e.g. for pure fcc metals, in which cell structures do develop. If there were an independent athermal contribution to the flow stress, there would have to be an intercept of the data such that the rate sensitivity vanishes at a finite stress.

6.2.3. Stress Relaxation and Unloading Effects

Stress relaxation after plastic deformation exhibits an ever decreasing rate of stress decrease, which is a consequence of the rapid decrease of strain rate with stress. The temptation is great to extrapolate to a finite stress level at which the rate would vanish. This interpretation does not hold up under more sophisticated evaluations.⁹⁹ In fact, in the best tests at the lowest rates,¹¹³ the rate sensitivity increases again (due to the increasing influence of dynamic recovery at very low rates, in our interpretation).

When the specimen is unloaded rapidly but partially, one observes what might be called 'negative stress relaxation', and this terminology would lead one to believe in a finite stress where stress relaxation is zero, and to call this an 'internal stress level'. This is, again, a misleading interpretation. The response on both sides of this presumed internal stress level is by no means symmetrical (as one should expect in such an interpretation); indeed, the 'negative relaxation' has all the marks of a transient effect: the behavior is not reproducible. We would interpret all 'negative relaxation' as just another expression of unloading effects: the reverse motion of some of the last dislocations to have moved forward. This occurs primarily in the cell interiors, and while these are, indeed, under a back-stress (Section 6.2.2), this is not an additive contribution to forward flow.

Of all the parameters that have been called an 'internal stress', the 'relaxed level' is perhaps the least deserving one. Further support for

this opinion is to be seen in the fact that it typically depends on temperature and actually tends to be a constant fraction ($\sim 75\%$) of the current flow stress.

6.2.4. Power-law Creep

The steady-state relation between stress and strain-rate is often adequately described as a power law (Section 5.2.3). When a direct fit of the data gives a stress exponent of order 10, another description is often tempting: namely, to subtract an 'internal stress' from the applied stress to form an 'effective stress', which then only requires a smaller exponent—which is easier to explain. This is very common practice. It is justified when there are other, independent reasons to postulate a lower-limit stress, such as in two-phase materials.

There is one cause for such a threshold that is always legitimate, but almost never important (certainly not in our regime of interest): namely, the thermodynamic threshold.^{1,9} It represents the work necessary to provide whatever energy needs to be stored during plastic deformation. (See Section 2.1.6.)

6.2.5. The Flow-stress Plateau at Intermediate Temperatures

A diagram of flow stress versus temperature almost invariably exhibits three regions (Fig. 7): a fairly rapid decrease at low temperatures, a more-or-less constant flow stress at intermediate temperatures (at least when divided by an elastic modulus), and another decrease at temperatures in excess of, typically, half the melting point. The high-temperature drop is almost certainly associated with concurrent dynamic recovery and its associated rearrangement strains (Section 5.2.4). But if it were not there, perhaps the temptation would be less to regard the low and intermediate temperature regimes as separate. Both regimes together certainly fall qualitatively into the category of mechanisms (all those discussed here) that get harder and harder to activate as the stress decreases. Thus, the so-called plateau could be merely a *very* slowly decreasing stress (similar to the late part of a stress-relaxation test).

In materials that contain substitutional solutes, there are two other reasons for 'plateau'-like behavior: one is that here the activation energy may well be proportional to the reciprocal of the stress, which makes for a much slower decrease with temperature;²⁹ the other reason lies in an expected mobility of solutes which, in extreme cases, would actually give a 'hump' in the $\tau(T)$ diagram.

In conclusion, there are a number of at least qualitative explanations for the appearance of a more-or-less level flow stress at intermediate temperatures that do not require the introduction of a separate 'internal stress';³³ such a new parameter should only be used when there are other reasons for needing it. The easiest example of such another reason, as pointed out repeatedly, is a failure of the Cottrell-Stokes law in the direction that extrapolates to a zero rate- or temperature-sensitivity at a finite stress (Section 3.1.7).

6.2.6. Conclusion

While there are many causes for true, tensorial internal stresses in real materials, most of these affect only transient behavior, but not the macroscopic flow stress. The latter is an appropriate quantity to characterize the deformed state of the material, using as many parameters as necessary. The first of these parameters is always the mechanical threshold: the flow stress back-extrapolated to zero temperature, *below* which most deformation happens. Sometimes, but by no means always, there is, in addition, a lower threshold or (scalar) 'athermal stress', which may superpose with other contributions to the plastic resistance in a linear or nonlinear fashion. Finally, and probably only in two-phase materials, there is a true long-range, tensorial internal back-stress (in the deforming matrix) which must be subtracted from the applied stress tensor before insertion into the yield criterion: this is the only legitimate cause for 'kinematic hardening'.

7. APPLICATION

As was pointed out in the Introduction, the most important lesson to be learned from materials science is that one cannot expect a single set of constitutive relations to hold for all materials and conditions. A worthy long-range aim of constitutive-relations development is to establish a systematic procedure by which it can be ascertained which class of materials and which regime of conditions a particular foreseen application falls into.

In this article, we have focussed on one such class and one such regime—even though we believe it to be a very broad and important case: uniform plastic deformation in (primarily single-phase) metals and alloys, at not too high or too low a temperature, at not too high or too low a strain rate, at not too small or too large strains. We have

stated in the Introduction the expected range of these variables under which (quasi-static) dislocation motion and accumulation dominate the behavior. In Section 7.1 we will outline some experimental tests that can be used to establish whether one indeed is in this regime, and what observations might warn one of special effects. Then, in Section 7.2, we will summarize the set of constitutive relations that appear to be most useful for this 'interest space'.

7.1. Diagnostics

7.1.1. *Stress/Strain/Strain-rate Test*

A very large amount of information can be derived from an analysis of the simplest test of all: a single tension test in which the whole stress/strain curve is recorded.[†] The only additional variable that must be introduced to assure this pay-off is the strain rate: it should be abruptly changed many times during the test. Good interpretability is ensured by using one of two procedures: either change the strain-rate up and down (by about a factor of 10) regularly every 3% strain or so; or make the slow rate the 'master' rate at which most of the straining is done and make occasional excursions to a higher strain rate and back. The amount of strain in any excursion must be large enough to have passed through any transient behavior, and the data acquisition system must be sufficiently sensitive to record the subtle changes. Even in materials in which the rate sensitivity is negligible in magnitude, a rate change can serve as a potent diagnostic tool.

First, an evaluation without rate changes. Assess whether there is a yield drop—in which case ignore that part of the stress/strain curve, since it relates to non-uniform deformation. Then see whether the slope of the load–elongation curve is reasonably constant for the first few percent strain, and of order $E/50$ (E is Young's modulus): then you are certainly within our 'interest space'. If it is much smaller (by an order of magnitude or so) you probably have a pre-deformed material, or perhaps an aged one—not all is lost.

If the initial plastic part of the stress/strain curve looks more parabolic than linear, there are two easy possibilities: either (at high temperatures) you are close to flow-stress saturation, which will be

[†] A compression test is just as useful, especially when not enough material is available, or when the ductility is quite limited. Teflon or some other lubrication can usually assure homogeneous deformation, which can be checked by inspection of the shape of the deformed specimen.

obvious (and you may wish to treat everything before as a transient); or you have a material in which a second phase plays a significant role. In the latter case, plot σ^2 vs. ϵ to see whether you get an essentially straight initial part. In this case, Section 7.1.2 becomes important. (We always assume materials of cubic lattice structure; in non-cubic polycrystals, there may be an extended region of elasto-plastic transition, also linked to high internal stresses.¹¹⁴)

Next check the appearance of any transients upon an *increase* in strain-rate. If they are less than about 0.5% long, ignore them and extrapolate through them: they are due either to machine effects¹¹⁵ or to some adjustments in the dynamic dislocation structure.¹⁰ If the stress/strain curve after any such short transient is accurately parallel to the one at the lower strain rate, you have the easiest case of all: a single state parameter (which might as well be the strain in this case). On the other hand (which is more likely at larger strains), if you have a higher-than-normal strain-hardening rate for more than 1% strain, it is probably due to a significant influence of dynamic recovery, and thus more complicated kinetics: Sections 5.2.4 through 5.2.6 apply, and you must decide whether this transient is important for your application or may be extrapolated through.

Now for a quantitative evaluation of the rate sensitivity. First, assess whether $\Delta \ln \sigma / \Delta \ln \dot{\epsilon}$, the way you measure it (using the applied stress, no 'effective stress'), is smaller or larger than 0.03. If larger, you are probably measuring a steady-state rate-sensitivity $1/n$ (eqn (5.7)), or at least one that is significantly influenced by dynamic recovery. For use in the following analysis, it would be better to try a more 'instantaneous' evaluation, i.e. a shorter back-extrapolation. If a truly iso-structural value is between, say, 0.03 and 0.01, the glide-controlling obstacle is probably quite small (a solute acting individually, or the 'Peierls stress'); in this case, nonlinear elastic effects are likely to be important and the parallelism of yield surface and plastic potential may well be violated (Section 4.2.1).

As a next step, plot $m\sigma = \Delta\sigma / \Delta \ln \dot{\epsilon}$, at various strains, versus the flow stress at that strain: the 'Haasen plot' (Section 3.1.7). Warning: if the slope of this line is negative, and m itself is not negative yet, it may become so at higher strains or temperatures or concentrations, and lead to jerky flow.³⁸

If the data in the Haasen plot extrapolate to the origin, the Cottrell-Stokes law is obeyed and a single state parameter suffices to describe the monotonic-kinetic behavior. If the plot gives a fairly

straight line that does *not* extrapolate to the origin, two flow-stress contributions superpose linearly; one may be 'athermal' if the line extrapolates to a finite stress at zero rate sensitivity. In the latter case, there may be a kinematic-hardening component (to be ascertained by the tests under Section 7.1.2). In general, both contributions may be thermally activated.

Finally, assess the rate sensitivity of the ultimate tensile strength: a special case of $1/n$, as defined in eqn (5.7). If it is less than about 0.1, the formalism outlined in Section 5.2.3 can be used, without the need of introducing any 'pre-exponential' stress exponent, and most likely without a real influence of diffusion mechanisms.⁸⁵ When $1/n > 0.1$ (and the temperature is larger than half the melting point) more caution is advised.

Last but not least, look at the specimen, for any evidence of grossly non-uniform deformation, or of anisotropic deformation. If the aspect ratio of the cross-section of a tensile sample has changed, the material was not isotropic at the start of deformation; in fact, it did not even have the symmetry of the tensile test: a fiber in the tensile axis. For this purpose, an initially cubic compression specimen is even more instructive, since shear strains can be seen that would not be expected on the basis of the test symmetry. When any initial anisotropy is discovered, the yield function to be used is unknown—and not likely to be well represented by a von Mises assumption.

7.1.2. Stress Reversal and Recovery

When there is suspicion of a kinematic hardening component (for example, because of initial parabolic hardening, or a positive stress intercept on the Haasen plot), two additional tests are advisable. The easiest is an annealing test, at a temperature where recrystallization does not occur (Section 6.1.5): is the flow stress upon reloading under the old conditions less than 80% of the one reached previously? Or, more to the point, does the new stress/strain curve fail, by a substantial margin and after many percent strain, to join up with the extrapolation of the previous one? (Actually, it is the behavior in a diagram of Θ vs. σ that gives the best information.¹¹⁰) Then, a kinematic hardening component, of the order of this failure, may be appropriate.

Secondly, if possible, a Bauschinger test would be instructive: if the behavior is of a transient nature, 'normal' after a few percent strain, a kinematic hardening component is *not* indicated (Section 6.1.4).

7.1.3. Temperature Dependence

This is the hardest test to execute; it depends on the application whether, and in which range, it is worthwhile. At very low temperatures (say, <50 K), one must assess whether dislocation inertia may be important;²⁶ this is indicated when the flow stress falls below the back-extrapolation from higher temperatures. At somewhat higher, yet subenvironmental temperatures, a strong decrease of stress with increasing temperature is probably due to an influence of the Peierls stress, and is possibly connected to a non-normality of the strain-increment on the yield surface.

At intermediate temperatures (typically at room temperature and above), one may wish to investigate the nature of an apparent stress plateau (Section 3.1.4), especially if the rate sensitivity indicated a negative trend in the Haasen plot: then, dynamic strain-aging and its instabilities may occur under slightly changed conditions. In solution-hardened alloys in general, one may wish to plot σ vs. $T \cdot \sigma$, to see whether this provides a reasonable description.²⁹ If the plateau is associated with a positive stress intercept in the Haasen plot, an investigation of the temperature dependence of the strain-rate sensitivity can be quite instructive.¹¹⁶

Finally, at high temperatures (say, $>T_m/2$), the treatment of the yield stress becomes complicated (it being influenced by concurrent dynamic recovery), but near-steady-state behavior may be assessed reasonably well. Here again, it becomes important to study the temperature dependence of the rate sensitivity—except now the one in steady state, $1/n$. An instructive plot is the stress exponent n versus $\mu b^3/kT$: it is likely to extrapolate to a finite value (around 4) at infinite temperatures (eqn (5.9)). For values near there, some superposition of mechanisms is probably appropriate. But so long as n depends (in a continuous fashion) on temperature, this is evidence for a stress-dependent activation energy, and thus for inapplicability of diffusion-only mechanisms.

7.2. Constitutive Relations

We will now give, as an example, one set of constitutive relations for cubic metals or alloys fulfilling the following assumptions (using the symbol $\in \{ \quad, \quad \}$ to mean 'in the range of'). The specified ranges are not necessarily restrictive, as we have discussed other cases in the foregoing; we merely mean that within these ranges we should be

comfortably within the scope of the explicit or implicit assumptions made.

(a) Only macroscopic flow will be treated; loading and unloading transients will be ignored. This may be stated as

$$\Delta \epsilon \gg \Delta \sigma / E \quad (E \text{ is the elastic modulus})$$

(b) There is no tensorial internal back-stress at this macroscopic flow level:

$$\alpha_{ij} \ll \sigma_{ij}$$

(Otherwise, perhaps all equations still hold if α_{ij} is subtracted from σ_{ij} , whenever the stress appears as a tensor.)

(c) The 'instantaneous' rate sensitivity of the flow stress (as measured by the applied equivalent stress):

$$m \equiv \left. \frac{\partial \ln \sigma}{\partial \ln \dot{\epsilon}} \right|_{T, \bar{\epsilon}} = \epsilon \{0, 0.01\}$$

(d) The strain-hardening rate (again measured by macroscopic variables):

$$\Theta \equiv \left. \frac{\partial \sigma}{\partial \epsilon} \right|_{T, \dot{\epsilon}} = \epsilon \{\sigma/2, E/50\}$$

The upper limit, $E/50$, again suggests that there is no kinematic-hardening component. The lower limit, $\sigma/2$, is set beyond the 'ultimate tensile strength' (UTS), for which $\Theta = \sigma$; but it is chosen at the safe side of the point where the strain-hardening behavior typically changes in a qualitative manner (at about $\Theta \approx \sigma/4$ or perhaps $E/500$). In creep, the minimum creep rate under constant load is included; the regime much beyond this level does require more scrutiny.

(e) The rate sensitivity of the UTS (the reciprocal of the stress exponent of the minimum creep rate at constant load):

$$\frac{1}{n_1} \equiv \left. \frac{\partial \ln \sigma_{\text{UTS}}}{\partial \ln \dot{\epsilon}} \right|_T < 0.1$$

(Even $1/6$ probably works.) This indicates a strong influence of *dynamic* recovery (and less, if any, of diffusion).

7.2.1. Multiaxial Relations

For simplicity, we only summarize the relations for the case that fully constrained conditions apply to all grains: an upper bound, and a good approximation for equi-axed grains. Then, as far as the material element is concerned, all strain increments are locally prescribed.† Split up into direction and magnitude, they may be labelled

$$d\epsilon_v = \epsilon_v^0 d\epsilon \quad (7.1)$$

(We now drop all bars for averages and double bars for prescribed components.)

The deviatoric stress becomes, in principle:

$$\sigma'_\mu = P_{\mu\nu} \epsilon_v^0 M \tau \quad (7.2)$$

In practice, as we have elaborated in Section 4.2.5, the matrix $P_{\mu\nu}$ is known only for isotropy under the von Mises assumption, when it is $\delta_{\mu\nu}$. But plastic anisotropy is of major qualitative importance for any description of multiaxial behavior, and no simple description of its effects exists as yet. For a quantitative treatment, it is imperative that $P(\epsilon^0)$ be tabulated for the entire relevant subspace.

The values $P_{\mu\nu}$ for each ϵ^0 can, in principle, be obtained in three ways: by experimental yield surface determinations under all expected conditions; by experimental texture measurements followed by a polycrystal calculation; or by a polycrystal calculation *ab initio*, assuming the initial texture to be known (e.g. isotropic). It is not impossible that the last method becomes feasible in the not-too-distant future. For now, only cases near isotropy can be explicitly treated: say, for 20–30% strain from an isotropic situation—or the trivial case of proportional loading of a specimen that already has (at least) the symmetry of the test.

Finally, to describe the evolution of τ , one will need the microscopic equivalent strain increment

$$d\Gamma = M d\epsilon \quad (7.3)$$

7.2.2. Kinetics and Evolution

With respect to the kinetics of flow and strain-hardening, one may adopt the same point of view as for the multiaxial part of the constitutive relations: they are too complicated to expect a sufficiently

† If, in fact, some components of *stress* are locally prescribed, this requires an iterative procedure.

accurate description in terms of functions—tables are needed. The principal use of the results of physical plasticity is then a specification of what the normalized parameters are that should be tabulated or plotted against each other. This is what we shall do in the following; but we will also repeat the simple functions that have given a reasonable description in many cases, for the regime specified at the beginning of Section 7.2.

The flow stress may have more than one component; however, one cannot make use of this in a phenomenologically sound way unless one has an operational way to separate the components, such as by using the Haasen plot of rate sensitivity (Section 3.1.7), or the Hall–Petch plot of grain-size dependence. The appearance of ‘plateaus’ in flow-stress vs. temperature or vs. time diagrams is not sufficient cause.

Any one component of flow stress is likely to obey a law of the kind (eqn (3.9))

$$\frac{\tau}{\mu(T, p)} = \frac{\hat{\tau}}{\mu_0} \cdot s \left(\frac{kT}{\mu b^3} \ln \frac{\dot{\gamma}_0}{M\dot{\epsilon}} \right) \quad (7.4)$$

where (eqn (3.8) and using x for the argument of s above)

$$\begin{aligned} s(x) &\approx (1 - (x/g_0)^{2/3})^2 \\ \dot{\gamma}_0 &\approx 10^8 \text{ s}^{-1} \end{aligned} \quad (7.4')$$

and $\hat{\tau}/\mu_0$ is obtained by back-extrapolation to $s = 1$, i.e. $T = 0$.

The dependence of the state parameter $\hat{\tau}$ on history parameters such as strain is not explicitly stated; it must be given in differential form. Most easily, it is specified directly by the variation of τ (not $\hat{\tau}$) with strain at a standard T and $\dot{\epsilon}$, in which case the value of s in eqn (7.4) should, strictly, appear as a factor in the following equations.

The strain-hardening rate (eqns (5.3) and (7.3)) appears to have the general form (eqn (5.6))

$$\theta = \theta_a - \theta_r \left(\frac{\tau - \tau_0}{\tau_s} \right) \quad (7.5)$$

where the general function may have the special form

$$\begin{aligned} \theta_r(r) &\approx \theta_a r \\ \theta_a &\approx \mu/200 \end{aligned} \quad (7.5')$$

τ_0 is the yield stress, and τ_s is a scaling parameter that obeys the

relation (eqn (5.8'))

$$\ln \frac{\tau_s}{\mu} = \ln \frac{\tau_{s_0}}{\mu_0} + S \left(\frac{kT}{\mu b^3} \ln \frac{\dot{\gamma}_{s_0}}{M\dot{\epsilon}} \right) \quad (7.6)$$

where the general function $S(x)$ approximately obeys (eqn (5.8))

$$\begin{aligned} S(x) &\approx x/g_s \\ \dot{\gamma}_{s_0} &\approx 10^5 \text{ s}^{-1} \end{aligned} \quad (7.6')$$

τ_{s_0}/μ_0 is obtained by back-extrapolation to $T = 0$, and g_s follows from the measured slope of the plot.

When eqns (7.5') and (7.6') apply, one can extrapolate to steady-state creep with a stress exponent n that is independent of stress and strain-rate, but inversely proportional to temperature. At higher temperatures, eqn (7.6') is not found to be sufficiently accurate, and a useful expression for n becomes (eqn (5.9))

$$n = n_0 + \frac{\mu b^3}{kT} g_s \quad (7.7)$$

with $n_0 \approx 4$.

Transient creep is best described by the deceleration parameter (eqns (5.10), (5.13), (5.14))

$$\delta = \frac{\Theta}{m\sigma} \quad (7.8)$$

However, it is at present not clear whether the strain-hardening rate measured in the 'macro-flow' way (past the 'long transient'; Section 5.2.6) is in fact relevant near steady state, or whether some measure of this strain-hardening transient is more appropriate. In general, one should use a differential description such as outlined in Section 5.2.5.

When the relations (7.4), (7.5) and (7.6) hold, but not their primed counterparts, a tabular description may be necessary—but one has gained a great deal from applying the physical principles appropriate for this regime: e.g. temperature and strain rate appear essentially as a single parameter, $kT \ln \dot{\epsilon}$, properly normalized by material constants; the strain-hardening rate depends on stress only, and the stress-scaling parameter depends on the strain-rate/temperature combination. Thus, one needs only three two-dimensional plots or tables, not the whole four-dimensional space spanned by τ , θ , T and $\dot{\epsilon}$.

7.2.3. Conclusion

We have presented a sample, a tentative example, of the procedure we envisage for future determinations of constitutive behavior: a series of diagnostic tests, proceeding from simple to more complex if necessary; and a set of normalized parameters between which, for the regime of interest, relations (in functional or tabular form) are to be found. This technique needs much development work.

8. SUMMARY AND RECOMMENDATIONS

Article of Faith

The current behavior of a material depends only on its current state, which is embodied in its current structure (microstructure, substructure). History variables such as time and strain can enter the constitutive relations only in differential form: describing the evolution of the state. The current rate of evolution is one aspect of current behavior, determined by the current state. Rearrangements of the structure upon loading are also entirely determined by the structure in the unloaded state.

Elastic-Plastic Transition

The elastic-plastic transition is very sharp; deviatoric elastic strains are negligible with respect to plastic ones except in the very vicinity of the yield stress. The constitutive behavior using a yield or flow stress defined by back-extrapolation of the strain-hardening curve is much simpler and more reproducible than the regime in which both plastic and elastic distortions must be taken into account, and the simpler description does capture the most important features of plasticity.

Yield Surface for Rate-sensitive Materials

The concept of a yield criterion is useful even for rate-sensitive materials, especially when (as usual) the rate sensitivity is low. Overstress formulations may be appropriate at very high strain-rates ($>10^5 \text{ s}^{-1}$). In the normal range, thermal activation dominates rate sensitivity, and it *lowers* the yield stress from the rate-independent (zero temperature) limit, the 'mechanical threshold', which is a primary parameter of the dislocation theory of flow.

Normality Flow Rule

The strain-increment direction is normal to the yield surface if, and only if, the state of the material, as it affects the plastic resistance, does not depend on the state of stress. This is true for most metals in the normal range of applications, but is expected to be violated under conditions of non-negligible nonlinear-elastic effects (e.g. in very-high-strength materials, at very high pressures, and in polymers). In any case, a plastic potential exists and provides a useful description in terms of the relation between strain *rates* and (total) stresses (rather than stress rates).

Plastic Anisotropy

The shape of the yield surface is strongly influenced by the crystallographic texture of the material. Even at mild textures, it departs significantly from a von Mises 'sphere' and 'oval' shapes: extensive flat regions and sharp (though not necessarily discontinuous) ridges and vertices develop, which are very important for plastic-stability considerations. It is unlikely that a useful *functional* relation for such realistic yield surfaces will be found; numerical descriptions appear mandatory. Appropriate subspaces have been identified in which, for a given texture symmetry, the yield surface must be derived (from experiment or simulation).

Hardening Rules

The evolution of the yield surface is also strongly influenced by the texture changes accompanying plastic deformation. These expected changes in shape are superposed on a general increase in size. 'Kinematic hardening' is significant only in two-phase materials, so long as the macroscopic flow stress is being described.

Kinetics of Flow and Evolution

The rate dependence of the flow stress is usually negligible, but it provides an important tool to identify deformation mechanisms; in particular whether more than one contribution to the flow stress can be operationally identified (such as an 'internal stress' in addition to the mechanical threshold). The rate dependence of strain hardening is more substantial and more important; it controls steady-state flow and, more generally, flow at a low strain-hardening rate (whether this be due to large prestrains or high temperature).

Strain Hardening

Strain hardening consists of an athermal and a dynamic-recovery part. The latter is primarily stress dependent, with a stress-scaling parameter that depends on temperature and strain rate. At very high strains, an additional, little-explored stage causes slow but steady hardening in many materials.

Transients

Noticeable transients in the strain-hardening behavior occur after changes such as rate-changes, stress reversal, and recovery. They last for about 3% strain. If they are significant for the particular application, an additional state parameter is necessary to describe the evolution of each type of transient.

Differential Constitutive Relations

An attractive alternative to constitutive descriptions in terms of (tentatively) identified state parameters and their evolution is an entirely differential description, in terms of external state variables only; in that case, the initial conditions are equivalent to state parameters, and the number and order of such differential equations must allow for the requisite number of initial conditions. A new formulation of this kind has been proposed that incorporates hardening transients.

Temperature Dependence

The most important temperature dependence is that of the steady-state limit stress, or other parameters characterizing dynamic recovery. In some cases at least, it obeys a single law over the entire temperature range up to at least $2/3$ of the melting point; changes in mechanism need not then be postulated. The activation energy is stress dependent, over this regime, and correspondingly the stress exponent depends on temperature in a continuous fashion.

The temperature dependence of the yield stress is mild by comparison, but in most cases it is not nil, so that an athermal, 'internal' stress need not be postulated as an additional parameter. Solute hardening, and especially solute mobility, play an important role in this regime.

Creep

The view is taken here that plasticity is inherently rate sensitive and thus indistinguishable (except for the boundary conditions) from

creep. It is, however, an important open question whether the creep behavior near steady state is correlated with regular strain hardening or with the prominent hardening transients in this regime.

Internal Stresses

There are reliable methods to test whether any macroscopic state parameter that fits any of the many meanings of this term needs to be introduced. This is expected in two-phase materials.

Normalized Variables

Various external variables and material parameters occur, according to crystal plasticity theory, in certain combinations only. This can reduce significantly the amount of information that need be collected. Some such combinations are: τ/μ , $(kT/\mu b^3) \ln(\dot{\epsilon}/\dot{\epsilon}_0)$ (where only $\dot{\epsilon}_0$ is a free parameter to which, however, the results are not sensitive), and θ/μ as well as $\theta/m\tau$, the deceleration rate in creep.

Functional versus Numerical Description

In general, we have come to the conclusion that functional relations will rarely be easy and accurate enough to describe the various dependences outlined above. The most important task of physical constitutive relations development is the identification of the smallest space in which the relationships must be specified. Then, a numerical description (tabular or diagrammatic) may be feasible. Inasmuch as most applications in which realistic material descriptions are needed are in numerical, computer-code form anyway, a numerical approach to materials behavior is not inopportune.

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